TD4 : Numerical elastoplasticity
Local aspects

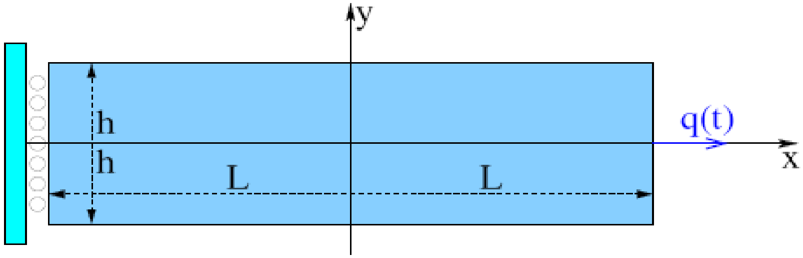
- Nonlinear problems: elastoplasticity (local aspects)

  == > Local integration of the elastoplastic constitutive law

  == > Radial return algorithm

- A simple example

- A Von Mises type model with nonlinear isotropic hardening



TD4: Numerical elastoplasticity
Local aspects

Equilibrium at instant t (weak form)

$$\int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{v}) d\Omega = \int_{S_T} \underline{T}^d . \underline{v} d\Gamma + \int_{\Omega} \underline{f}^d . \underline{v} d\Omega \qquad \forall\ \underline{v} \in C(0) \quad \forall t \in [0,T]$$

Kinematic conditions: $\underline{u} \in C(\underline{u}^d) \quad \left(\underline{u} = \underline{u}^d \text{ on } S_u\right)$

Constitutive equations

$$\underline{\underline{\sigma}}(t) = A : \left(\underline{\underline{\varepsilon}}(t) - \underline{\underline{\varepsilon}}^p(t)\right)$$

Elasticity

$$\dot{\underline{\underline{\varepsilon}}}^p(t) = \dot{\gamma} \frac{\partial f}{\partial \underline{\underline{\sigma}}}$$

Yield function

$$f\left(\underline{\underline{\sigma}}, \ldots\right) \le 0 \ ; \ \dot{\gamma} \ge 0 \, ; \ f\left(\underline{\underline{\sigma}}, \ldots\right) \dot{\gamma} = 0$$

normality

Yield function - plastic multiplier - complementarity

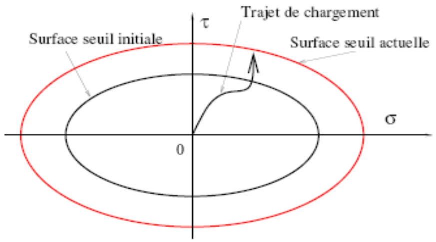
Initial condition: $\underline{\underline{\varepsilon}}^p(t=0) = \underline{\underline{0}}$

To be precised :
- *Elasticity law (A)*
- *Yield function (f)*
- *Hardening (isotropic, kinematic)*

TD4 : Von Mises – Isotropic hardening

Assumptions:

- linear isotropic elasticity
- Von Mises criterion; plastic rate respects normality law
- Isotropic hardening



Questions:

- For $f(\underline{\underline{\sigma}}, p) = \sigma^{eq} - R(p)$ prove $\dfrac{\partial f}{\partial \underline{\underline{\sigma}}} = \dfrac{3}{2}\dfrac{\underline{\underline{s}}}{\sigma^{eq}}$
- Prove $\dot{\gamma} = \dot{p}$
- Deduce the constitutive equations

$$\underline{\underline{\sigma}} = [3\kappa\mathcal{J} + 2\mu\mathcal{K}] : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^{\mathrm{P}}) = \kappa \mathrm{tr}(\underline{\underline{\varepsilon}}) + 2\mu(\underline{\underline{e}} - \underline{\underline{\varepsilon}}^{\mathrm{P}})$$ Elasticity

$$f(\underline{\underline{\sigma}}, p) = \sigma^{\mathrm{eq}} - R(p) \leq 0$$ Yield function

$$\dot{\underline{\underline{\varepsilon}}}^{\mathrm{P}} = \dot{p}\frac{\partial f}{\partial \underline{\underline{\sigma}}}(\underline{\underline{\sigma}}, p) = \frac{3\dot{p}}{2\sigma^{\mathrm{eq}}}\underline{\underline{s}}\,, \quad \dot{p} \geq 0\,, \quad \dot{p}(\sigma^{\mathrm{eq}} - R(p)) = 0$$ normality

$$\sigma^{\mathrm{eq}} = \sqrt{\tfrac{3}{2}}\|\underline{\underline{s}}\|$$

$$\underline{\underline{N}} = \sqrt{\frac{3}{2}}\frac{\underline{\underline{s}}}{\sigma^{\mathrm{eq}}}$$

$$\underline{\underline{s}} = \underline{\underline{\sigma}} - \frac{tr(\underline{\underline{\sigma}})}{3}\underline{\underline{1}}$$

$$\underline{\underline{e}} = \underline{\underline{\varepsilon}} - \frac{tr(\underline{\underline{\varepsilon}})}{3}\underline{\underline{1}}$$

$$\dot{p} = \sqrt{\frac{2}{3}\dot{\underline{\underline{\varepsilon}}}_p : \dot{\underline{\underline{\varepsilon}}}_p}$$

- Linear isotropic hardening: $R(p) = h\,p + \sigma_0$

TD4 : Numerical elastoplasticity

Equilibrium at instant t (weak form)

$$\int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{v}) d\Omega = \int_{S_T} \underline{T}^d . \underline{v} d\Gamma + \int_{\Omega} \underline{f}^d . \underline{v} d\Omega \qquad \forall \ \underline{v} \in C(0) \quad \forall t \in [0, T]$$

Kinematic conditions: $\underline{u} \in C(\underline{u}^d) \quad \left(\underline{u} = \underline{u}^d \text{ on } S_u\right)$

Constitutive equations

$$\underline{\underline{\sigma}} = [3\kappa \mathcal{J} + 2\mu \mathcal{K}] : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^{P}) = \kappa \operatorname{tr}(\underline{\underline{\varepsilon}}) + 2\mu(\underline{\underline{e}} - \underline{\underline{\varepsilon}}^{P})$$

Elasticity

$$f(\underline{\underline{\sigma}}, p) = \sigma^{eq} - R(p) \leq 0$$

Yield function

$$\dot{\underline{\underline{\varepsilon}}}^{P} = \dot{p} \frac{\partial f}{\partial \underline{\underline{\sigma}}}(\underline{\underline{\sigma}}, p) = \frac{3\dot{p}}{2\sigma^{eq}} \underline{\underline{s}}, \quad \dot{p} \geq 0, \quad \dot{p}(\sigma^{eq} - R(p)) = 0$$

normality

Algorithm: compute $\mathcal{S}_n \stackrel{\text{def}}{=} \{\underline{u}_n, \underline{\underline{\varepsilon}}_n, \underline{\underline{\varepsilon}}^{P}_n, \underline{\underline{\sigma}}_n \dots\}$ at every instant $t = t_n$

== > Incremental approach (step by step): knowing $\mathcal{S}_n$ and $(\underline{f}_{n+1}, \underline{u}^{D}_{n+1}, \underline{T}^{D}_{n+1})$, gives $\mathcal{S}_{n+1}$

TD4 : Radial return algorithm

Knowing $\mathcal{S}_n = \{\underline{u}_n, \underline{\underline{\varepsilon}}_n, \underline{\underline{\varepsilon}}^{\mathrm{P}}_n, \underline{\underline{\sigma}}_n\}$ (t=t_n) ; and $(\underline{f}_{n+1}, \underline{u}^{\mathrm{D}}_{n+1}, \underline{T}^{\mathrm{D}}_{n+1})$ (t=t_{n+1})

Equilibrium at instant t=t_{n+1} (weak form)

$$\int_\Omega \underline{\underline{\sigma}}_{n+1} : \underline{\underline{\varepsilon}}[\underline{w}]\,\mathrm{d}V = \int_\Omega \rho \underline{f}_{n+1} . \underline{w}\,\mathrm{d}V + \int_{S_T} \underline{T}^{\mathrm{D}}_{n+1} . \underline{w}\,\mathrm{d}S \qquad \forall \underline{w} \in \mathcal{C}(\underline{0}).$$

Knowing: $\mathcal{S}_n$ and $\underline{u}_{n+1}$ find $\underline{\underline{\sigma}}_{n+1}$

$$(\underline{u}_{n+1}, \mathcal{S}_n) \longrightarrow \underline{\underline{\sigma}}_{n+1} = \mathcal{F}(\underline{u}_{n+1}; \mathcal{S}_n)$$

TD4

Local treatment of the constitutive equations

Radial return algorithm

== >

Find $\underline{u}_{n+1} \in \mathcal{C}(\underline{u}^{\mathrm{DD}}_{n+1}), \quad \mathcal{R}(\underline{u}_{n+1}; \underline{w}, \mathcal{S}_n) = 0 \qquad \forall \underline{w} \in \mathcal{C}(\underline{0})$

$$\mathcal{R}(\underline{u}_{n+1}; \underline{w}, \mathcal{S}_n) = \int_\Omega \mathcal{F}(\underline{u}_{n+1}; \mathcal{S}_n) : \underline{\underline{\varepsilon}}[\underline{w}]\,\mathrm{d}V - \int_\Omega \rho \underline{f}_{n+1} . \underline{w}\,\mathrm{d}V - \int_{S_T} \underline{T}^{\mathrm{D}}_{n+1} . \underline{w}\,\mathrm{d}S.$$

TD5

TD4 : Radial return algorithm

Knowing: $\mathcal{S}_n$ and $\underline{u}_{n+1}$ find $\underline{\underline{\sigma}}_{n+1}$

$$(\underline{u}_{n+1}, \mathcal{S}_n) \longrightarrow \underline{\underline{\sigma}}_{n+1} = \mathcal{F}(\underline{u}_{n+1}; \mathcal{S}_n)$$

Local treatment of the constitutive equations

Radial return algorithm

Constitutive equations

$$\underline{\underline{\sigma}} = \kappa \mathrm{tr}(\underline{\underline{\varepsilon}}) + 2\mu(\underline{\underline{e}} - \underline{\underline{\varepsilon}}^{\mathrm{P}})$$

$$f(\underline{\underline{\sigma}}, p) = \sigma^{\mathrm{eq}} - R(p) \leq 0 \qquad \dot{\underline{\underline{\varepsilon}}}^{\mathrm{P}} = \frac{3\dot{p}}{2\sigma^{\mathrm{eq}}}\underline{\underline{s}} \qquad \dot{p} \geq 0 \qquad \dot{p}(\sigma^{\mathrm{eq}} - R(p)) = 0$$

Local Local treatment of the constitutive equations:

+ Temporal discretization: $t_{n+1} = t_n + \Delta t$

+ Numerical integration $\dot{\underline{\underline{\varepsilon}}}^{\mathrm{P}}(t) \approx \frac{1}{\Delta t}[\underline{\underline{\varepsilon}}^{\mathrm{P}}_{n+1} - \underline{\underline{\varepsilon}}^{\mathrm{P}}_{n}] \qquad (t_n \leq t \leq t_{n+1})$

== > Discretization of the constitutive equations

+ Radial return algorithm:

For a given $\Delta\varepsilon_n$ ($\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$), compute S_{n+1} at t_{n+1}

TD4 : Constitutive equations Discrete version

$$\underline{\underline{\sigma}} = \kappa \operatorname{tr}(\underline{\underline{\varepsilon}}) + 2\mu(\underline{\underline{e}} - \underline{\underline{\varepsilon}}^{\mathrm{P}})$$

$$f(\underline{\underline{\sigma}}, p) = \sigma^{\mathrm{eq}} - R(p) \leq 0 \qquad \dot{\underline{\underline{\varepsilon}}}^{\mathrm{P}} = \frac{3\dot{p}}{2\sigma^{\mathrm{eq}}}\underline{\underline{s}} \qquad \dot{p} \geq 0 \qquad \dot{p}(\sigma^{\mathrm{eq}} - R(p)) = 0$$

Numerical integration : $t_n \Longrightarrow t_{n+1} = t_n + \Delta t$

1- Deduce the discretization of the constitutive equations at instant t_{n+1} *(implicit scheme)*

(P)

$$\underline{\underline{\sigma}}_{n+1} = \underline{\underline{\sigma}}_n + \kappa Tr(\Delta\underline{\underline{\varepsilon}}_n)\underline{\underline{1}} + 2\mu\left(\Delta\underline{\underline{e}}_n - \Delta\underline{\underline{\varepsilon}}_n^p\right)$$

$$\mathrm{f}(\underline{\underline{\sigma}}_{n+1}, p_n + \Delta p_n) = \underline{\underline{\sigma}}_{n+1}^{eq} - \left(\sigma_0 + h\left(p_n + \Delta p_n\right)\right) \leq 0$$

$$\Delta\underline{\underline{\varepsilon}}_n^p = \frac{3\underline{\underline{S}}_{n+1}}{2\underline{\underline{\sigma}}_{n+1}^{eq}}\Delta p_n = \sqrt{\frac{3}{2}}\Delta p_n \underline{\underline{N}}_{n+1} \quad ; \quad \Delta p_n \geq 0 \quad ; \quad \Delta p_n \mathrm{f}(\underline{\underline{\sigma}}_{n+1}, p_n + \Delta p_n) = 0$$

Constitutive equations (discrete version)

2- Radial return algorithm ?

Solve problem (P): for a given $\Delta\varepsilon_n$ $(\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n)$*, compute* S_{n+1} *at instant* $t = t_{n+1}$

TD4 : Radial return algorithm

Elastic prediction:

$$\underline{\underline{\sigma}}_{n+1}^{elas} = \underline{\underline{\sigma}}_n + \kappa Tr(\Delta \underline{\underline{\varepsilon}}_n)\underline{\underline{1}} + 2\mu \Delta \underline{\underline{e}}_n$$

$$\underline{\underline{S}}_{n+1}^{elas} = \underline{\underline{S}}_n + 2\mu \Delta \underline{\underline{e}}_n$$

$$\mathrm{f}_{n+1}^{elas} = \mathrm{f}(\underline{\underline{\sigma}}_{n+1}^{elas}, p_n) = \sigma_{n+1}^{eq} - (\sigma_0 + hp_n)$$

Convexity of f ==>

$$\mathrm{f}_{n+1}^{elas} = \mathrm{f}(\underline{\underline{\sigma}}_{n+1}^{elas}, p_n) \geq \mathrm{f}(\underline{\underline{\sigma}}_{n+1}, p_n + \Delta p_n)$$

if $\mathrm{f}_{n+1}^{elas} < 0$ ==>

$$\begin{cases} \underline{\underline{\sigma}}_{n+1} = \underline{\underline{\sigma}}_{n+1}^{elas} \\ \Delta p_n = 0 \ ; \ \Delta \underline{\underline{\varepsilon}}_n^p = 0 \end{cases}$$

Otherwise, solve

if $\mathrm{f}_{n+1}^{elas} > 0$ ==>

$$\begin{cases} \underline{\underline{\sigma}}_{n+1} = \underline{\underline{\sigma}}_{n+1}^{elas} - 2\mu \Delta \underline{\underline{\varepsilon}}_n^p \\ \underline{\underline{S}}_{n+1} = \underline{\underline{S}}_{n+1}^{elas} - 2\mu \Delta \underline{\underline{\varepsilon}}_n^p \end{cases}$$

with

$$\begin{cases} \Delta \underline{\underline{\varepsilon}}_n^p = \Delta p_n \sqrt{\frac{3}{2}} \underline{\underline{N}}_{n+1} \\ \underline{\underline{S}}_{n+1} = \sqrt{\frac{2}{3}} \sigma_{n+1}^{eq} \underline{\underline{N}}_{n+1} \end{cases}$$

$$f(\underline{\underline{\sigma}}_{n+1}, p_n + \Delta p_n) = 0$$

TD4 : Radial return algorithm

Plastic correction:

if $f_{n+1}^{elas} > 0$ $\quad \begin{cases} \underline{\underline{\sigma}}_{n+1} = \underline{\underline{\sigma}}_{n+1}^{elas} - 2\mu\Delta\underline{\underline{\varepsilon}}_n^p \\ \underline{\underline{S}}_{n+1} = \underline{\underline{S}}_{n+1}^{elas} - 2\mu\Delta\underline{\underline{\varepsilon}}_n^p \end{cases}$ with $\begin{cases} \Delta\underline{\underline{\varepsilon}}_n^p = \Delta p_n \sqrt{\frac{3}{2}}\underline{\underline{N}}_{n+1} \\ \underline{\underline{S}}_{n+1} = \sqrt{\frac{2}{3}}\sigma_{n+1}^{eq}\underline{\underline{N}}_{n+1} \end{cases}$

$$f(\underline{\underline{\sigma}}_{n+1}, p_n + \Delta p_n) = 0$$

We have $\underline{\underline{S}}_{n+1} = \underline{\underline{S}}_{n+1}^{elas} - 2\mu\Delta\underline{\underline{\varepsilon}}_n^p$ We obtain $\underline{\underline{S}}_{n+1}^{elas} = \left(\sqrt{\frac{2}{3}}\sigma_{n+1}^{eq} + 2\mu\sqrt{\frac{2}{3}}\Delta p_n\right)\underline{\underline{N}}_{n+1}$

and, as a consequence $\underline{\underline{N}}_{n+1} = \underline{\underline{N}}_{n+1}^{elas}$

It gives $\sqrt{\frac{2}{3}}\left(\sigma_{n+1}^{eq} + 3\mu\Delta p_n\right)\underline{\underline{N}}_{n+1} = \underline{\underline{S}}_{n+1}^{elas}$ and finally $\sigma_{n+1}^{eq} = \sigma_{n+1}^{eq^{elas}} - 3\mu\Delta p_n$

We now write the consistency condition: $f(\underline{\underline{\sigma}}_{n+1}, p_n + \Delta p_n) = \sigma_{n+1}^{eq} - R(p_n + \Delta p_n) = 0$

$==>$ solve with respect to Δp_n

$$\sigma_{n+1}^{eq^{elas}} - 3\mu\Delta p_n - R(p_n + \Delta p_n) = 0$$

For linear isotropic hardening:

$$\Delta p_n = \frac{f_{n+1}^{elas}}{h + 3\mu}$$

To prove !

TD4 : Radial return algorithm - RR_VonMises.m

Summary: radial return algorithm

(a) Compute $\underline{\underline{s}}^{elas}_{n+1} = \underline{\underline{s}}_n + 2\mu\Delta\underline{\underline{e}}_n$ (elastic predictor), then $\underline{\underline{\sigma}}^{elas}_{n+1}$ and $\sigma^{elas,eq}_{n+1}$;

(b) Compute $f(\underline{\underline{\sigma}}^{elas}_{n+1}, p_n)$ and **test** :

- **If $f(\underline{\underline{\sigma}}^{elas}_{n+1}, p_n) \leq 0$ (elastic evolution)**, solution given by :

$$\underline{\underline{\sigma}}_{n+1} = \underline{\underline{\sigma}}^{elas}_{n+1}, \quad \underline{\underline{\varepsilon}}^P_{n+1} = \underline{\underline{\varepsilon}}^P_n, \quad p_{n+1} = p_n \qquad \text{(END)}$$

- **If $f(\underline{\underline{\sigma}}^{elas}_{n+1}, p_n) > 0$ (elastoplastic evolution) :**

  (i) Solve $\sigma^{elas,eq}_{n+1} - 3\mu\Delta p_n - R(p_n + \Delta p_n) = 0$ for Δp_n;

  (ii) Compute the increment of plastic strains

$$\Delta\underline{\underline{\varepsilon}}^P_n = \frac{3\Delta p_n}{2\sigma^{elas,eq}_{n+1}}\underline{\underline{s}}^{elas}_{n+1};$$

  (iii) Update variables :

$$\underline{\underline{\varepsilon}}^P_{n+1} = \underline{\underline{\varepsilon}}^P_n + \Delta\underline{\underline{\varepsilon}}^P_n, \qquad p_{n+1} = p_n + \Delta p_n$$
$$\underline{\underline{\sigma}}_{n+1} = \underline{\underline{\sigma}}_n + \kappa\,\mathrm{tr}(\Delta\underline{\underline{\varepsilon}}_n)\underline{\underline{1}} + 2\mu(\underline{\underline{e}}_n - \Delta\underline{\underline{\varepsilon}}^P_n) \qquad \text{(END)}$$

Plane strain $\Delta\underline{\underline{\varepsilon}}_{33} = 0$

$$\text{Deps} = [\Delta\varepsilon_{11}; \Delta\varepsilon_{22};\ 2\Delta\Delta_{12}]$$
$$\mathrm{Tr}(\Delta\underline{\varepsilon}) = \Delta\varepsilon_{11} + \Delta\varepsilon_{22} + 0$$
$$\text{De} = [\text{De}_{11}; \text{De}_{22}; \text{De}_{33}; \text{De}_{12}] = [\Delta\varepsilon_{11}; \Delta\varepsilon_{22}; 0; \Delta\varepsilon_{12}] - \frac{1}{3}\mathrm{Tr}(\Delta\underline{\varepsilon})[1;1;1;0]$$

- Linear isotropic hardening:

$$R(p) = h\,p + \sigma_0 \qquad \Delta p_n = \frac{f^{elas}_{n+1}}{h + 3\mu}$$

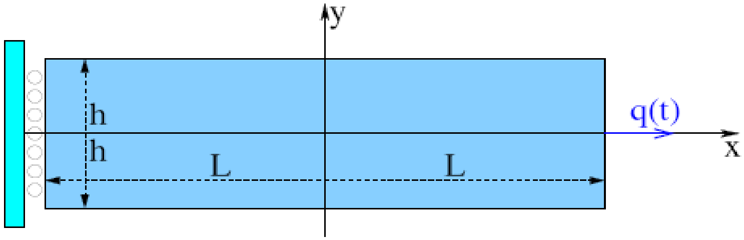
Matlab code: **all-elements**/RR_VonMises.m

Output: $(\Delta p_n, \underline{\underline{\sigma}}_{n+1})$ — Input: Elastic parameters (E, ν), Plastic parameters (h, σ_0), State at $t=t_n$ $(\underline{\underline{\sigma}}_n, p_n)$, Strain increment $(\Delta\underline{\underline{\varepsilon}}_n)$

$$(\Delta p_n, \underline{\underline{\sigma}}_{n+1}) = RR\_VonMises(E, \nu, h, \sigma_0, \underline{\underline{\sigma}}_n, p_n, \Delta\underline{\underline{\varepsilon}}_n)$$

```
function [Dp,sigma_new]=RR_VonMises(E,nu,H,sigma0,sigma,p,Deps)

mu=E/(2*(1+nu));
kappa=E/(3*(1-2*nu));
vl=[1 1 1 0]';

trDeps=Deps(1)+Deps(2);                              % trace of incremer
De=[Deps(1:2); 0; Deps(3)/2]-1/3*trDeps*vl;          % full 3D deviator
sigma_elas=sigma+kappa*trDeps*vl+2*mu*De;            % elastic predictio
trsigma=sum(sigma_elas(1:3));                        % volumetric part
s_elas=sigma_elas-1/3*trsigma*vl;                    % deviatoric elast
sigeq_elas=sqrt(1.5*...                              % equivalent devia
          (s_elas(1)^2+s_elas(2)^2+ ...
           s_elas(3)^2+2*s_elas(4)^2));
f_elas=sigeq_elas-H*p-sigma0;
if(f_elas>0)                                         % if plastic proce
  Dp=f_elas/(3*mu+H);                                % increment of plas
  sigeq_new=sigeq_elas-3*mu*Dp;                      % new equivalent st
  Depsp=3/2*Dp*s_elas/sigeq_elas;                    % increment over st
       sigma_new=sigma_elas-2*mu*Depsp;
else                                                 % elseif elastic pr
  Dp=0;
  sigma_new=sigma_elas;                              % new total stress
end
```

TD4 : Example *(see strip_plast.m)*



$$u_x(L, y) = q(t) \quad T_y(L, y) = 0$$
$$u_x(-L, y) = T_y(-L, y) = 0$$
$$T_x(x, \pm h) = T_y(x, \pm h) = 0$$

Exact solution available:

+ plane strain
+ homogeneous solution
  - ε_{xx}, ε_{yy} (other 0)
  - σ_{xx}, σ_{zz} (other 0)

Material: homogeneous, elastic, perfectly plastic (R'(p)=0)

Compute ε_{yy}, σ_{xx}, σ_{zz} such that

Kinematic admissibility

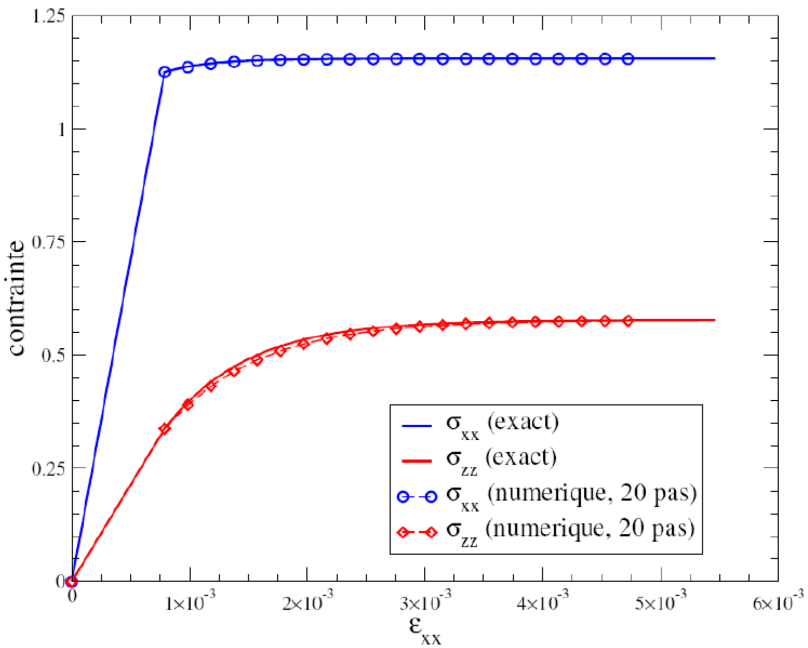
== > $\varepsilon_{xx} = q/2L$

Constitutive relation

=== > radial return algorithm

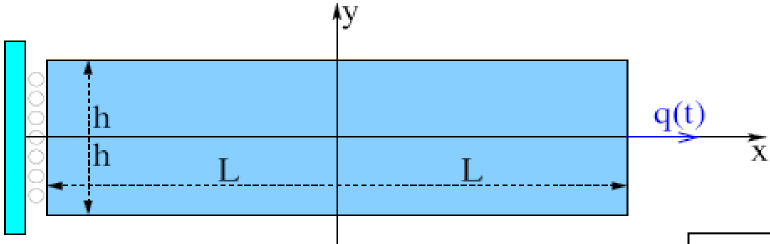
Equilibrium

== > $\sigma_{yy} = 0$



TD4 : A simple example

Algorithm (strip_plast.m)



$\underline{\underline{\sigma}}_n, \underline{\underline{\varepsilon}}_n, p_n$ known

Plane strain

$$A = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

Temporal discretization (loading increment): Δq

find $\underline{\underline{\sigma}}_{n+1}, \underline{\underline{\varepsilon}}_{n+1} = \underline{\underline{\varepsilon}}_n + \Delta\underline{\underline{\varepsilon}}^n, p_{n+1} = p_n + \Delta p_n$

Initialization

$\Delta\underline{\underline{\varepsilon}}^{n,0} = \delta\underline{\underline{\varepsilon}}^{n,0}$

$\delta\varepsilon_{xx}^{n,0} = \Delta q / 2L$ (imposed)

$\delta\varepsilon_{yy}^{n,0} = -\delta\varepsilon_{xx}^{n,0} A(1,2)/A(2,2)$ such as $(\underline{\underline{\sigma}}^0)_{yy} = (A:\delta\underline{\underline{\varepsilon}}^{n,0})_{yy} = 0$ (*choice*)

Radial return algo $==>$ $\underline{\underline{\sigma}}_{n+1}^{(0)} = \mathcal{F}(\Delta\underline{\underline{\varepsilon}}_n^{(0)}, \mathcal{S}_n)$ **but $\sigma^{(0)}_{yy}$ may not be zero !**

while $\sigma_{yy}^{n+1,k} \neq 0$

$\Delta\underline{\underline{\varepsilon}}^{n,k+1} = \Delta\underline{\underline{\varepsilon}}^{n,k} + \delta\underline{\underline{\varepsilon}}^{n,k}$

$\delta\varepsilon_{xx}^{n,k} = 0$ (imposed)

$\delta\varepsilon_{yy}^{n,k} = -(\sigma_{yy}^{n+1,k} + A(1,2)\delta\varepsilon_{xx}^{n,k})/A(2,2) = -\sigma_{yy}^{n+1,k}/A(2,2)$

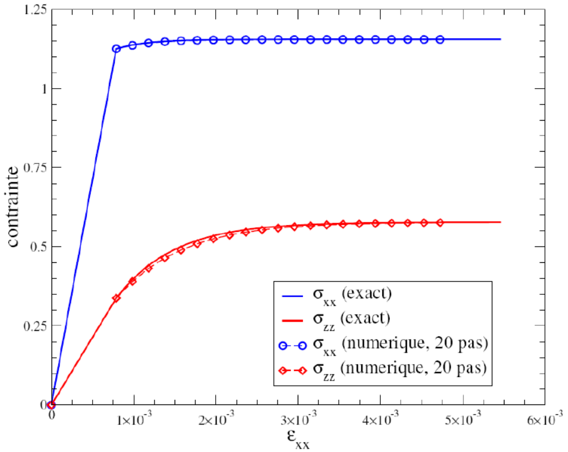
such as $(\underline{\underline{\sigma}}^{n+1,k} + A:\delta\underline{\underline{\varepsilon}}^{n,k})_{yy} = 0$ (*choice*)

Radial return algo $==>$ $\underline{\underline{\sigma}}_{n+1}^{(k)} = \mathcal{F}(\Delta\underline{\underline{\varepsilon}}_n^{(k)}, \mathcal{S}_n)$

Strip_plast.m

Iterate until convergence

TD4 : A simple example



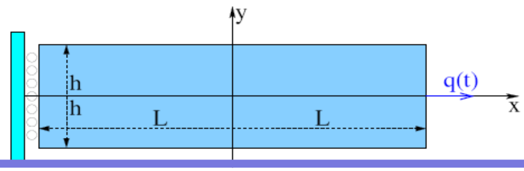
$$[A] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

Example : Strip_plast.m

```
L=1;
E=100;
nu=.1;
sigma0=1;
H=0;
q=[0:.002:.1];

%%%%%%%%%%%%%%%%%%%%%%%%
% Pre-processor
%%%%%%%%%%%%%%%%%%%%%%%%

numstep=length(q);
p=0;
sigma=zeros(4,1);
sigma_old=sigma;
output=zeros(numstep,2);
toll=1.d-4;
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% History analysis
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for istep=2:numstep,
  Dq=q(istep)-q(istep-1);
  iter=0;
  resid=1;
  Dp=0;
  Deps=zeros(3,1);
  while resid > toll,
    iter=iter+1;
    if iter==1
      Deps=Dq/(2*L)*[1 -nu/(1-nu) 0]';
    else
      Depsyy=-sigma_new(2)*(1+nu)* ...
                      (1-2*nu)/(E*(1-nu));
      Deps=Deps+[0 Depsyy 0]';
    end
    [Dp,sigma_new]=RR_VonMises(E,nu, ...
                   H,sigma0,sigma,p,Deps);
    resid=abs(sigma_new(2));
  end
  p=p+Dp;
  sigma=sigma_new;
  output(istep,:)=[sigma(1) sigma(3)];
end
```

Ex: A Von Mises type model with nonlinear isotropic hardening



1- Repeat the analysis with the yield function

$$f(\underline{\underline{\sigma}}, p) = \sigma^{eq} - \left(\sigma_0 + Hp^{1/M}\right)$$

2- Include modifications in *RR_VonMisesNL.m*

3- Simulate load path q(t): from 0 up to q_{fin} = L/100

Geometry : L = 0,05m
Material parameters (Acier 316L) :

E= 200 000 Mpa
ν = 0.3
σ_0 = 138 Mpa
H = 430 Mpa
M = 4.55

Other Yield functions …

Drucker-Prager

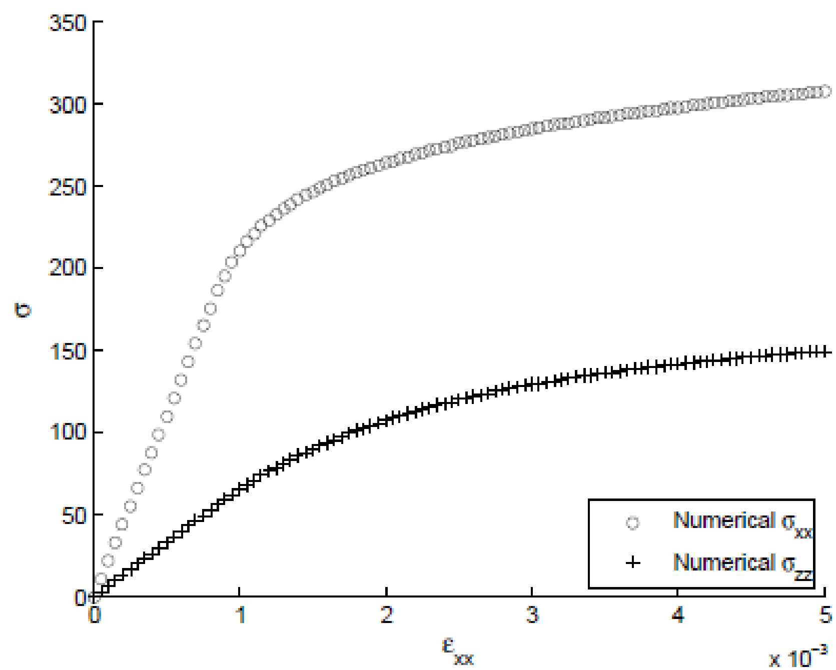
$$f = \sigma^{eq} - 3\alpha Tr(\underline{\underline{\sigma}}) - H\gamma - \sigma_0$$

Kinematic hardening

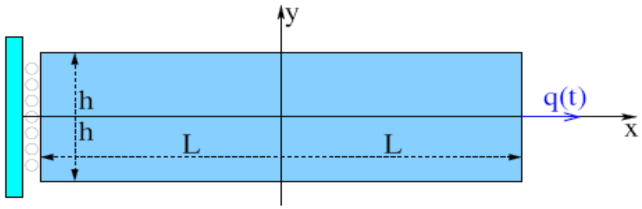
$$f(\underline{\underline{\sigma}}, \underline{\underline{X}}) = \left(\underline{\underline{\sigma}} - \underline{\underline{X}}\right)^{eq} - \sigma_0$$

$$\dot{\underline{\underline{X}}} = 2C\dot{\underline{\underline{\varepsilon}}}^p$$

use $\underline{\underline{\alpha}} = \underline{\underline{S}} - \underline{\underline{X}}$



DM: A Von Mises type model with nonlinear isotropic hardening



1- Repeat the analysis with the yield function

$$f(\underline{\underline{\sigma}}, p) = \sigma^{eq} - \left(\sigma_0 + Hp^{1/M}\right)$$

2- Include modifications in *RR_VonMisesNL.m*

3- Prove the initial yield limit q_0 is $q_0 = \dfrac{\sigma_0 2L(1-\nu^2)}{E\sqrt{1-\nu+\nu^2}}$.

4- Simulate load path q(t) (cycle)

1 : q(t) = 0	up to	q(t)= $2q_0$
2 : q(t) = $2q_0$	up to	q(t)= - $4q_0$
3: q(t) = - $4q_0$	up to	q(t)= 0

5- Simulate several cycles

6- Plot σ_{xx} in function of ε_{xx}

Geometry : L = 5 cm
Material parameters (Acier 316L) :

E= 200 000 Mpa
ν = 0.3
σ_0 = 138 Mpa
H = 430 Mpa
M = 4.55

Send by mail :
- Modified matlab code
  - *RR_VonMisesNL.m*
  - *Strip_plast.m*
- 2 pages (max) of comments and analysis + results (figures)