Contact mechanics and elements of tribology

Lecture 5.
Computational Contact Mechanics

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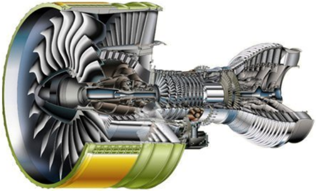
@ Centre des Matériaux
February 8, 2016

Outline

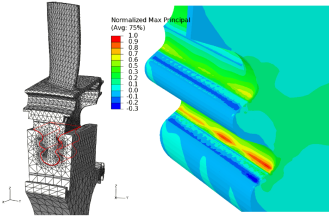
- Introduction
- Governing equations
- Optimization methods
- Resolution algorithm
- Examples

Industrial and natural contact problems

1. Assembled parts, e.g. engines



Aircraft's engine GP 7200
`www.safran-group.com`



[1] M. W. R. Savage
J. Eng. Gas Turb. Power, 134:012501 (2012)

Industrial and natural contact problems

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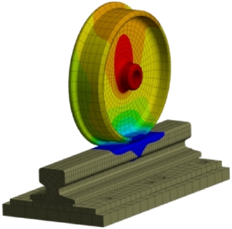
2. Railroad contacts

High speed train TGV www.sncf.com



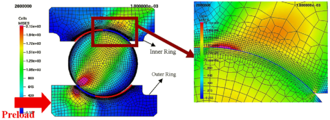
Wilde/ANSYS wildeanalysis.co.uk

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings

SKF

Bearings
www.skf.com



[1] F. Massi, J. Rocchi, A. Culla, Y. Berthier Mech. Syst. Signal Pr., 24:1068-1080 (2010)

Industrial and natural contact problems

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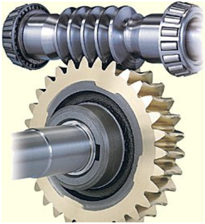
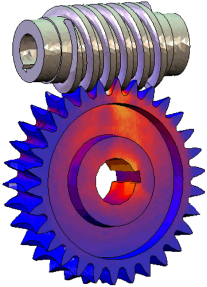


Helical gear www.tpg.com.tw



www.mscsoftware.com

Industrial and natural contact problems

1. Assembled parts, e.g. engines
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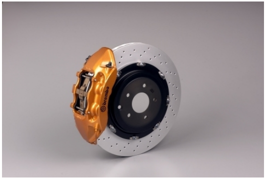
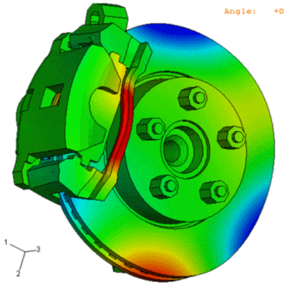
4. Breaking systems



Assembled breaking system

www.brembo.com



www.mechanicalengineeringblog.com

Industrial and natural contact problems

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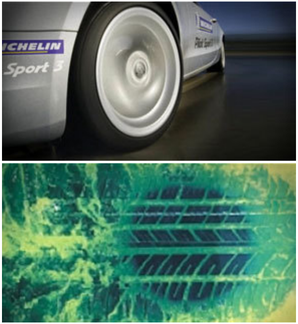
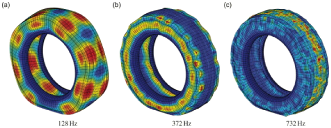
5. Tire-road contact



Tire-road contact www.michelin.com



[1] M. Brinkmeier, U. Nackenhorst, S. Petersen, O. von Estorff, J. Sound Vib., 309:20-39 (2008)

Industrial and natural contact problems

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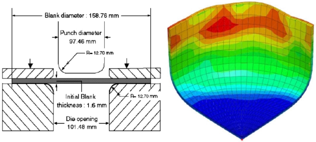
5. Tire-road contact
6. Metal forming

Deep drawing www.thomasnet.com



[1] G. Rousselier, F. Barlat, J. W. Yoon Int. J. Plasticity, 25:2383-2409 (2009)

Industrial and natural contact problems

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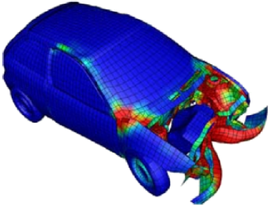
6. Metal forming
7. Crash tests

Crash-test `www.porsche.com`



[1] O. Klyavin, A. Michailov, A. Borovkov
`www.fea.ru`

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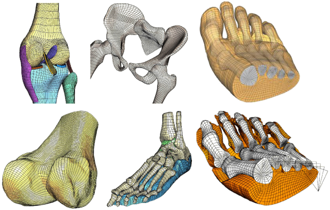
7. Crash tests
8. Biomechanics

Human articulations
www.sportssupplements.net



J. A. Weiss, University of Utah
Musculoskeletal Research Laboratories

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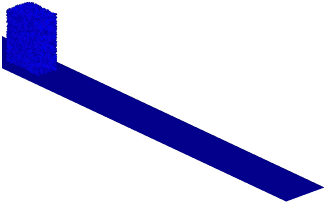
9. Granular materials

Sand dunes www.en.wikipedia.org



E. Azema et al, LMGC90

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8. Biomechanics
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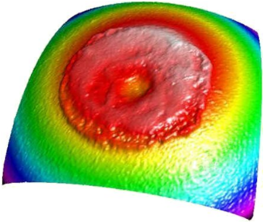
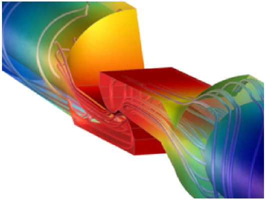
10. Electric contacts



Damage at electric contact zone
www.taicaan.com



Simulation of electric current
www.comsol.com

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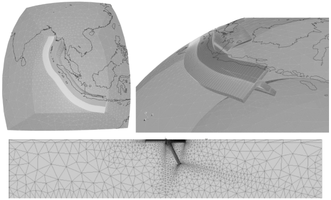
9. Granular materials
10. Electric contacts
11. Tectonic motions

San-Andreas fault, by M. Rightmire
www.sciencedude.ocregister.com



[1] J.D. Garaud, L. Fleitout, G. Cailletaud Colloque CSMA (2009)

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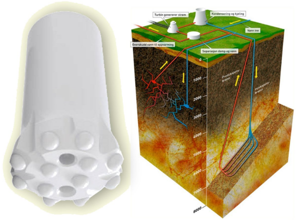
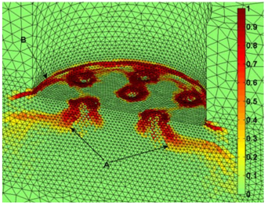
12. Deep drilling



Drill Bit tool **RobitRocktools**; *extraction of geothermal energy* (**SINTEF, NTNU**)



[1] T. Saksala, Int. J. Numer. Anal. Meth. Geomech. (2012)

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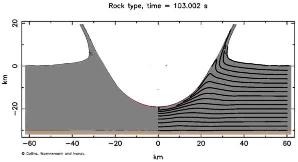
12. Deep drilling
13. Impact and fragmentation

Impact crater, Arizona
www.MrEclipse.com *et* maps.google.com



Simulation of formation of Copernicus crater
Yue Z., Johnson B. C., et al. Projectile remnants in central peaks of lunar impact craters. Nature Geo 6 (2013)

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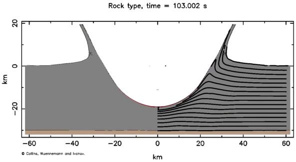
14. etc.

Impact crater, Arizona
www.MrEclipse.com et maps.google.com



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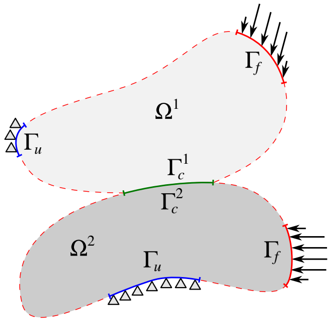
Equilibrium and contact conditions

- **Balance of momentum**

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0 & \text{on } \Gamma_f \\ \underline{u} = \underline{u}_0 & \text{on } \Gamma_u \\ \;\; ? & \text{on } \Gamma_c \end{cases}$$

- **Frictionless contact conditions *(intuitive)***

  1. No penetration
  2. No adhesion
  3. No shear transfer



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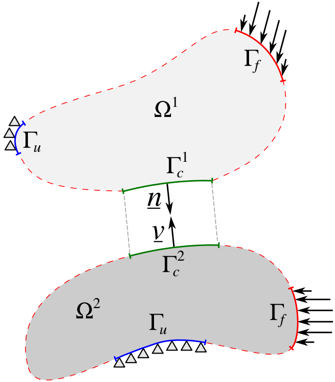

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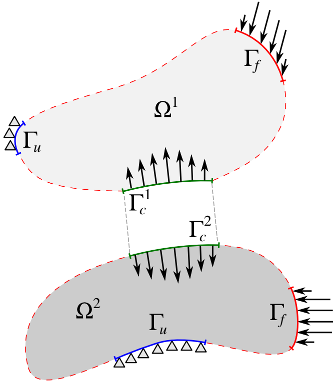

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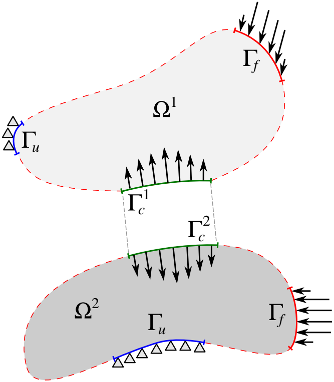

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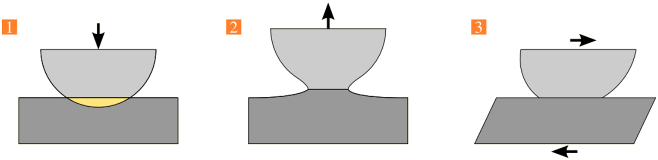

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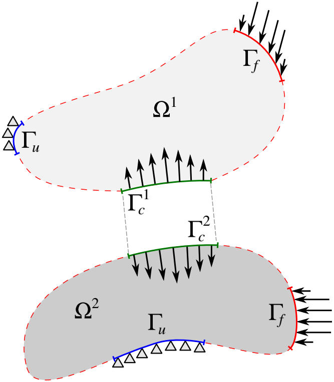

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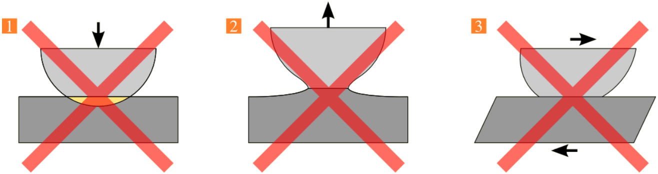
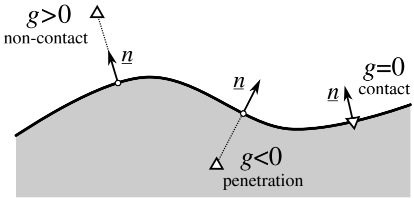


- **Gap function** g
  - gap = – penetration
  - asymmetric function
  - defined for
    - separation $g > 0$
    - contact $g = 0$
    - penetration $g < 0$
  - governs normal contact
- **Master and slave split**

  Gap function is determined for all slave points with respect to the master surface



Gap between a slave point and a master surface

Gap function

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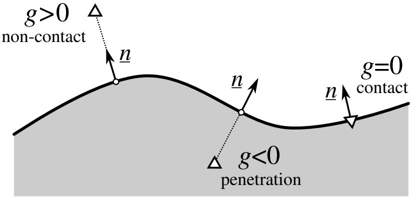


Gap between a slave point and a master surface

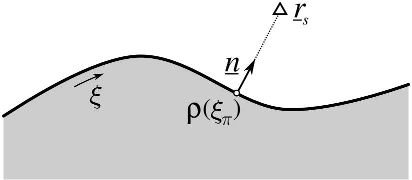
- **Master and slave split**

  Gap function is determined for all slave points with respect to the master surface

- **Normal gap**

$$g_n = \underline{n} \cdot \left[\underline{r}_s - \underline{\rho}(\xi_\pi) \right],$$

$\underline{n}$ is a unit normal vector, $\underline{r}_s$ slave point, $\underline{\rho}(\xi_\pi)$ projection point at master surface



Definition of the normal gap

Consider existence and uniqueness 

Frictionless or normal contact conditions

- **No penetration**
  Always non-negative gap

  $$g \geq 0$$

- **No adhesion**
  Always non-positive contact pressure

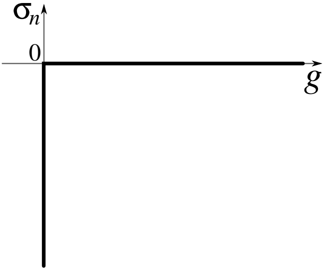
  $$\sigma_n^* \leq 0$$

- **Complementary condition**
  Either zero gap and non-zero pressure, or non-zero gap and zero pressure

  $$g\,\sigma_n = 0$$

- **No shear transfer** *(automatically)*

  $$\underline{\sigma}_t^{**} = 0$$



Scheme explaining normal contact conditions

$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$

$\underline{\sigma}_t^{**} = \underline{\underline{\sigma}} \cdot \underline{n} - \sigma_n \underline{n} = \underline{n} \cdot \underline{\underline{\sigma}} \cdot \left(\underline{\underline{I}} - \underline{n} \otimes \underline{n}\right)$

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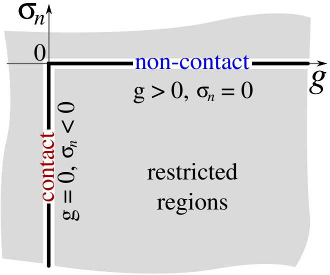

Improved scheme explaining normal contact conditions

$$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$$

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Frictionless or normal contact conditions

In **mechanics:**

Normal contact conditions

≡

Frictionless contact conditions

≡

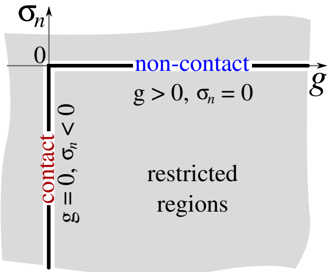
Hertz[1]*-Signorini*[2] *conditions*

≡

Hertz[1]*-Signorini*[2]*-Moreau*[3] *conditions*

also known in **optimization theory** as

Karush[4]*-Kuhn*[5]*-Tucker*[6] *conditions*



Improved scheme explaining normal contact conditions

$$g \geq 0, \qquad \sigma_n \leq 0, \qquad g\sigma_n = 0$$

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[1] **Heinrich Rudolf Hertz** (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.

[2] **Antonio Signorini** (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.

[3] **Jean Jacques Moreau** (1923) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.

[4] **William Karush** (1917–1997), [5] **Harold William Kuhn** (1925) American mathematicians,

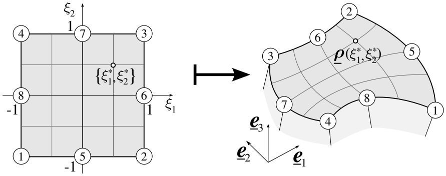
[6] **Albert William Tucker** (1905–1995) a Canadian mathematician.

Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1;1)$
- Mapping to real space

$$\underline{\rho}(\xi_1, \xi_2, t) = \sum_{i=1}^{8} N^i(\xi_1, \xi_2)\underline{\rho}^i(t)$$



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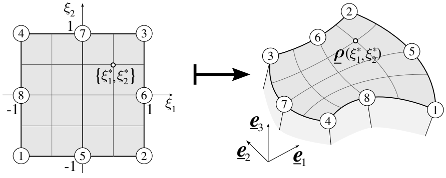
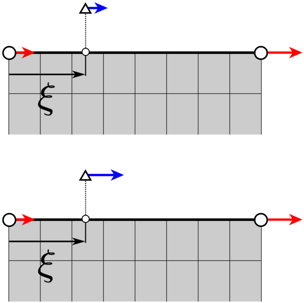


- **Tangential slip velocity** $\underline{v}_t$ must take into account:
  - only tangential component
  - relative rigid body motion
  - master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2}\dot{\xi}_2$$

where $\partial\underline{\rho}/\partial\xi_i$ are the tangent vectors of the local basis and ξ_i are the convective coordinates.



Relative slip between a slave point and a deformable master surface

Relative sliding: example

Consider a one-dimensional example:
P is a projection of A on segment BC.

$$x_P = \xi x_C + (1-\xi)x_B \qquad (1)$$

Velocity of the projection point

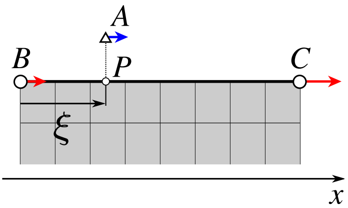
$$\dot{x}_P = \underbrace{\xi \dot{x}_C + (1-\xi)\dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B)\dot{\xi}}_{\frac{\partial x_P}{\partial \xi}\dot{\xi}}$$

Substract the velocity of point x_P for fixed ξ

$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B)\dot{\xi} = \frac{\partial x}{\partial \xi}\dot{\xi}$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left.\frac{\partial x}{\partial \xi}\right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip

Relative sliding: example

Consider a one-dimensional example:

P is a projection of A on segment BC.

$$x_P = \xi x_C + (1-\xi) x_B \qquad (1)$$

Velocity of the projection point

$$\dot{x}_P = \underbrace{\xi \dot{x}_C + (1-\xi)\dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B)\dot{\xi}}_{\frac{\partial x_P}{\partial \xi}\dot{\xi}}$$

Substract the velocity of point x_P for fixed ξ

$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B)\dot{\xi} = \frac{\partial x}{\partial \xi}\dot{\xi}$$

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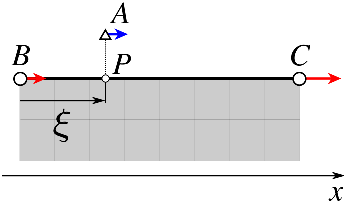
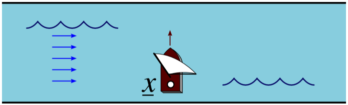


Example of a one-dimensional relative slip



Ship-river analogy

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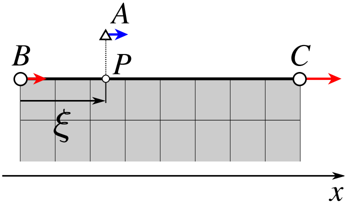
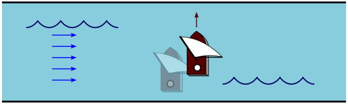


Example of a one-dimensional relative slip



Ship-river analogy

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Compute tangential slip increment

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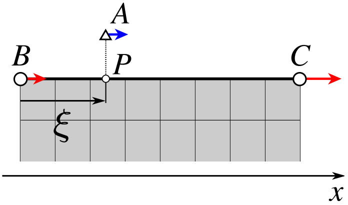
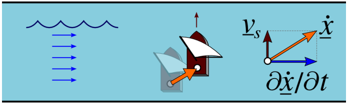


Example of a one-dimensional relative slip



Ship-river analogy

Li derivative: the change of a vector field along the change of another vector field

Amontons-Coulomb's friction

- **No contact** $g > 0, \ \sigma_n = 0$
- **Stick** $|\underline{v}_t| = 0$

  Inside slip surface/Coulomb's cone

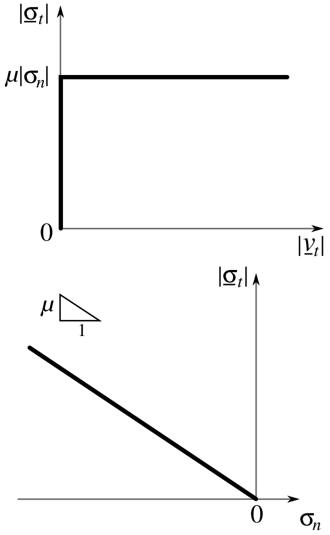
  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$
- **Slip** $|\underline{v}_t| > 0$

  On slip surface/Coulomb's cone

  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$
- **Complementary condition**

  Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

  $$|\underline{v}_t|\left(|\underline{\sigma}_t| - \mu|\sigma_n|\right) = 0$$



Scheme explaining frictional contact conditions

Amontons-Coulomb's friction

- **No contact** $g > 0, \ \sigma_n = 0$
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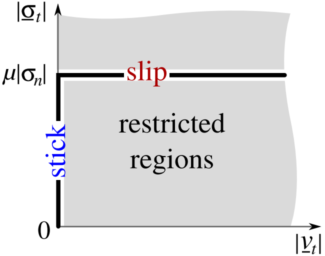
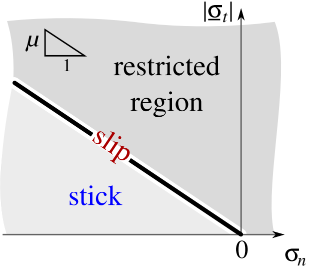




Improved scheme explaining frictional contact conditions

Amontons-Coulomb's friction

- **No contact** $g > 0, \ \sigma_n = 0$
- **Stick** $|\underline{v}_t| = 0$

  Inside slip surface/Coulomb's cone

  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$

- **Slip** $|\underline{v}_t| > 0$

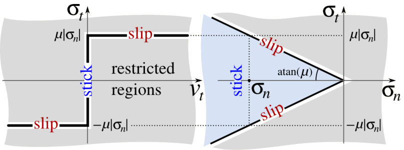
  On slip surface/Coulomb's cone

  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

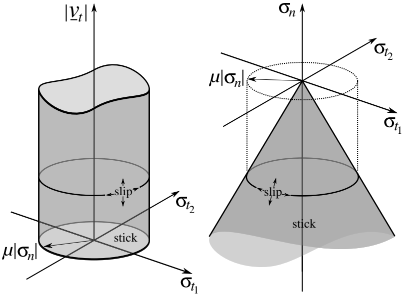
- **Complementary condition**

  Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

  $$|\underline{v}_t|\left(|\underline{\sigma}_t| - \mu|\sigma_n|\right) = 0$$



Scheme of 2D frictional contact

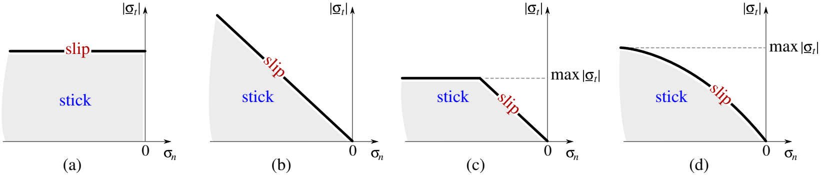


Scheme of 3D frictional contact

$$|\underline{v}_t| \geq 0, \qquad |\underline{\sigma}_t| - \mu|\sigma_n| \leq 0, \qquad |\underline{v}_t|\left(|\underline{\sigma}_t| - \mu|\sigma_n|\right) = 0$$

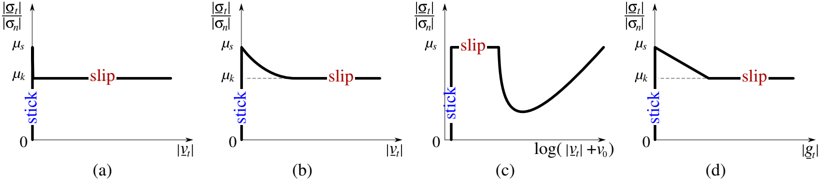
More friction laws

- **Static criteria**



(a) Tresca (b) Amontons-Coulomb (c) Coulomb-Orowan (d) Shaw

- **Kinetic criteria**



(a,b) velocity weakening (c) velocity weakening-strengthening
(d) Linear slip weakening

- μ_s static and μ_k kinetic coefficients of friction.

Rate and state friction and regularization

- **Rate and state friction law**
  - Rate $v_t = |\underline{v}_t|$ – relative slip velocity
  - State θ – ≈ internal time
  - Dieterich–Ruina–Perrin (1979, 83, 95)

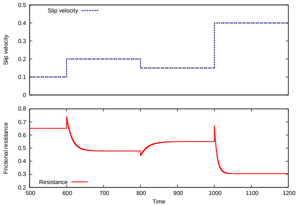
    Frictional resistance

    $$\sigma_t^c = |\sigma_n| \left[\mu_s + b\theta + a \ln(v_t/v_0)\right]$$

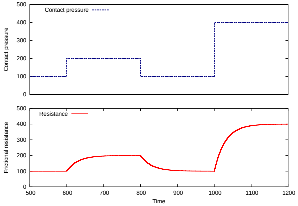
    Evolution of the state variable

    $$\dot{\theta} = -\frac{v_t}{L}\left[\theta + \ln\left(\frac{v_t}{v_0}\right)\right]$$

- **Prakash-Clifton friction law (1992,2000)**
  - Viscous type evolution of frictional resistance σ_t
  - $\dot{\sigma}_t = -\frac{v_t}{L}(\sigma_t + \mu\sigma_n)$



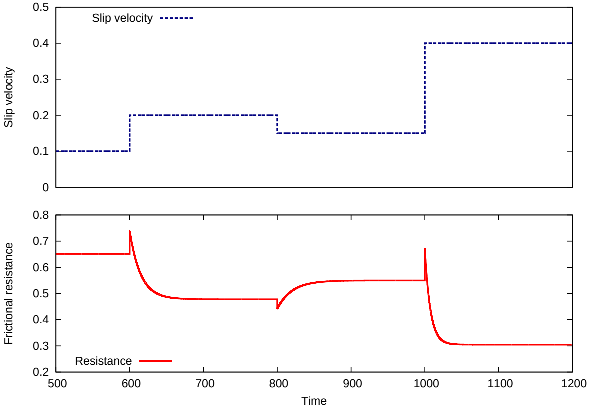
Rate and state friction law



Prakash-Clifton regularization

Rate and state friction and regularization

- **Rate and state friction law**



Rate and state friction and regularization

- **Rate and state friction law**
  - Rate $v_t = |\underline{v}_t|$ – relative slip velocity
  - State θ – $\approx$ internal time
  - Dieterich–Ruina–Perrin (1979, 83, 95)

    Frictional resistance

    $$\sigma_t^c = |\sigma_n| \left[\mu_s + b\theta + a\ln(v_t/v_0)\right]$$

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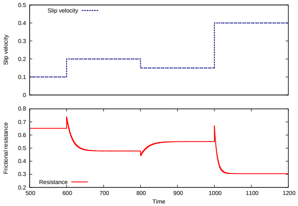
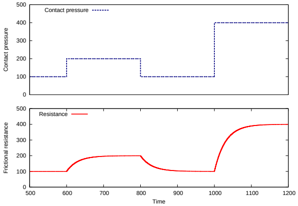


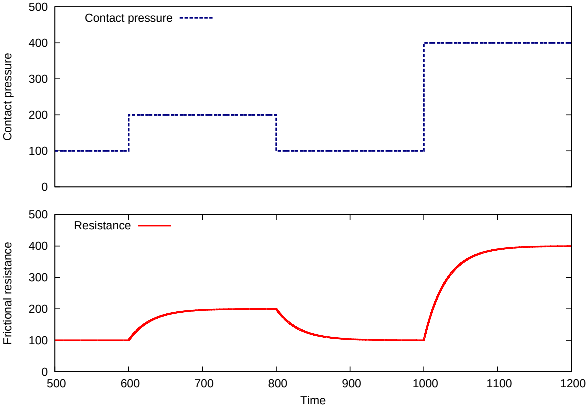
Rate and state friction law



Prakash-Clifton regularization

Rate and state friction and regularization

- Prakash-Clifton friction law (1992,2000)



Rate and state friction and regularization

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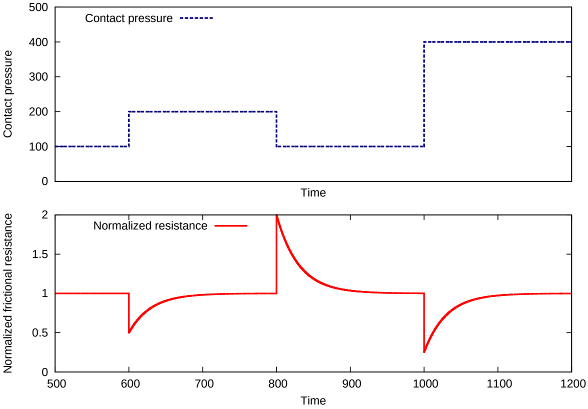
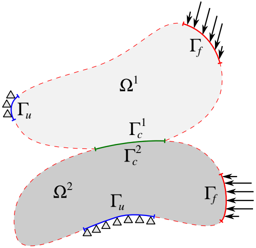


From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$



Two solids in contact

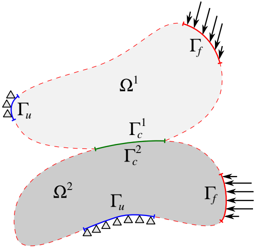
From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

- Balance of virtual works 

$$\boxed{\int_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{u} \, d\Gamma} + \int_{\Omega} \left[\underline{f}_v \cdot \delta\underline{u} - \underline{\underline{\sigma}} \cdot\cdot \, \delta\nabla\underline{u} \right] d\Omega = 0$$



Two solids in contact

From strong to weak form

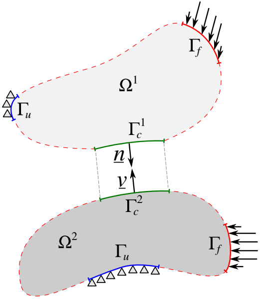
- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

- Balance of virtual works 

$$\boxed{\int\limits_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{u} \, d\Gamma} =$$

$$\boxed{\int\limits_{\overline{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\rho} \, d\overline{\Gamma}_c^1 + \int\limits_{\overline{\Gamma}_c^2} \underline{\nu} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{r} \, d\overline{\Gamma}_c^2} + \int\limits_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma_f$$



Two solids in contact

From strong to weak form

- Balance of momentum and boundary conditions

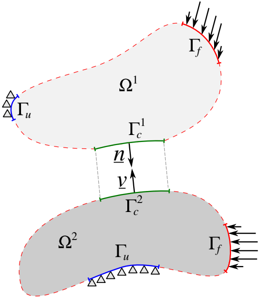
$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

- Balance of virtual works 

$$\boxed{\int\limits_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{u} \, d\Gamma} \Rightarrow$$

$$\boxed{\int\limits_{\bar{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\rho} \, d\bar{\Gamma}_c^1 + \int\limits_{\bar{\Gamma}_c^2} \underline{\nu} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{r} \, d\bar{\Gamma}_c^2} =$$

$$= \int\limits_{\bar{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta(\underline{\rho} - \underline{r}) \, d\bar{\Gamma}_c^1 = \int\limits_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}{}_t^T \delta \underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1$$



Two solids in contact

From strong to weak form

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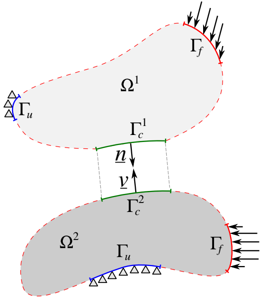


Two solids in contact

- Balance of virtual works 

$$\boxed{\int_{\partial\Omega} \underline{n}\cdot\underline{\underline{\sigma}}\cdot\delta\underline{u}\, d\Gamma} \Rightarrow$$

$$\boxed{\int_{\bar{\Gamma}_c^1} \underline{n}\cdot\underline{\underline{\sigma}}\cdot\delta\underline{\rho}\, d\bar{\Gamma}_c^1 + \int_{\bar{\Gamma}_c^2} \underline{\nu}\cdot\underline{\underline{\sigma}}\cdot\delta\underline{r}\, d\bar{\Gamma}_c^2} =$$

$$= \int_{\bar{\Gamma}_c^1} \underline{n}\cdot\underline{\underline{\sigma}}\cdot\delta(\underline{\rho}-\underline{r})\, d\bar{\Gamma}_c^1 = \int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}{}_t^T \delta \underset{\sim}{\xi}\right) d\bar{\Gamma}_c^1$$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot\cdot\,\delta\nabla\underline{u}\, d\Omega + \underbrace{\boxed{\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}{}_t^T \delta \underset{\sim}{\xi}\right) d\bar{\Gamma}_c^1}}_{\text{Contact term}} = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega$$

From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f_v} = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

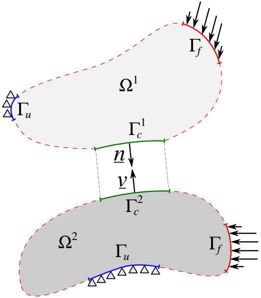
- Balance of virtual works 

$$\underbrace{\int_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u}\, d\Omega}_{\text{Change of the internal energy}} + \underbrace{\boxed{\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}{}_t^T \delta \underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1}}_{\text{Contact term}} =$$

$$\underbrace{\int_{\Gamma_f} \underline{\sigma_0} \cdot \delta \underline{u}\, d\Gamma}_{\text{Virtual work of external forces}} + \underbrace{\int_{\Omega} \underline{f_v} \cdot \delta \underline{u}\, d\Omega}_{\text{Virtual work of volume forces}}$$

- Functional space

$\underline{u} \in \mathbb{H}^1(\Omega)$ Hilbert space of the first order
and $\underline{u}$ satisfy boundary and **contact conditions**.



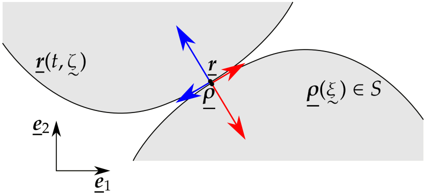
Two solids in contact

Towards variational inequality

- Contact term

$$\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n \, d\bar{\Gamma}_c^1 \leq 0$$



Contact configuration $\sigma_n \delta g_n = 0, \; \sigma_n \leq 0$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} \, d\Omega + \int_{\bar{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \, d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

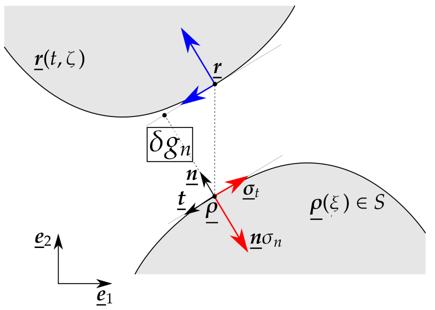
$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \text{ on } \Gamma_u, \; g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

Towards variational inequality

- Contact term

$$\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n \, d\bar{\Gamma}_c^1 \leq 0$$



Virtual change of the configuration

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} \, d\Omega + \int_{\bar{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \, d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

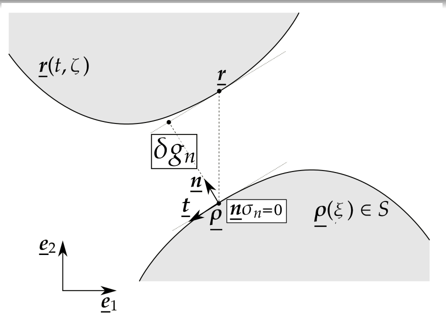
$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \text{ on } \Gamma_u,\; g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

Towards variational inequality

- Contact term

$$\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n \, d\bar{\Gamma}_c^1 \leq 0$$



Normal term in separation $\delta g_n > 0$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} \, d\Omega + \int_{\bar{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \, d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

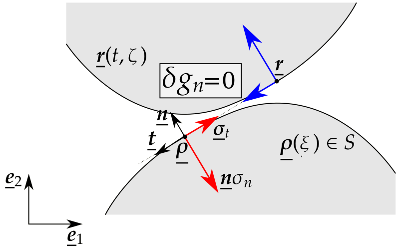
$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \text{ on } \Gamma_u, \; g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

Towards variational inequality

- Contact term

$$\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n \, d\bar{\Gamma}_c^1 \leq 0$$



Normal term in sliding $\delta g_n = 0$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} \, d\Omega + \int_{\bar{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \, d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \text{ on } \Gamma_u, \; g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

Back to variational equality (unconstrained)

- Constrained minimization problem

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} \, d\Omega + \int_{\bar{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi} \, d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \;\text{ on } \Gamma_u,\; g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

- Use optimization theory to convert to

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} \, d\Omega + \int_{\Gamma_c^1} \underbrace{\boxed{C(\sigma_n, \sigma_t, g_n, \underset{\sim}{\xi}, \delta \underline{u})}}_{\text{Contact term}^*} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

Unconstrained functional space $\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \;\text{ on } \Gamma_u \right\}$

Contact term* is defined on the *potential contact zone* Γ_c^1.

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method
- Lagrange multipliers method
- Augmented Lagrangian method

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- **Penalty method**
  - New functional

$$F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2} = F(\mathbf{x}) + \begin{cases} 0, & \text{if } g(\mathbf{x}) \geq 0 \quad \textit{non-contact} \\ \epsilon g^2(\mathbf{x}), & \text{if } g(\mathbf{x}) < 0 \quad \textit{contact} \end{cases}$$

    where ϵ is the penalty parameter.
  - Stationary point must satisfy

$$\nabla F_p(\mathbf{x}) = \nabla F(\mathbf{x}) + 2\epsilon \langle -g(\mathbf{x}) \rangle \nabla g(\mathbf{x}) = 0$$

  - Solution **tends** to the precise solution as $\epsilon \to \infty$

- Lagrange multipliers method
- Augmented Lagrangian method

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2}$
- **Lagrange multipliers method**
  - New functional called **Lagrangian**

$$\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})}$$

  - Saddle point problem

$$\min_x \max_\lambda \{\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})\} \longrightarrow \mathbf{x}^* \longleftarrow \min_{g(\mathbf{x})\geq 0} \{F(\mathbf{x})\}$$

  - Stationary point

$$\nabla_{\mathbf{x},\lambda} \mathcal{L} = \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_x g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0 \quad \text{need to verify} \quad \lambda \leq 0$$

- Augmented Lagrangian method

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2}$
- Lagrange multipliers method $\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})}$
- **Augmented Lagrangian method**

  [Hestnes 1969], [Powell 1969], [Glowinski & Le Tallec 1989], [Alart & Curnier 1991], [Simo & Laursen 1992]

  - New functional, augmented Lagrangian

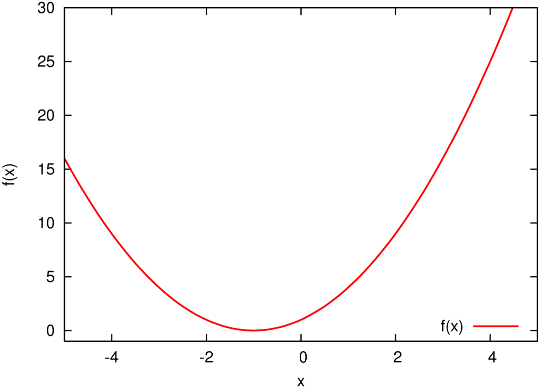
$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \ \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \ \textit{non-contact} \end{cases}$$

  - Stationary point

$$\nabla_{\mathbf{x},\lambda}\mathcal{L}_a = \begin{cases} \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_x g(\mathbf{x}) + 2\epsilon g(\mathbf{x}) \nabla g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0, & \textit{if contact} \\ \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) \\ -\frac{\lambda}{\epsilon} \end{bmatrix} = 0, & \textit{if non-contact} \end{cases}$$

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Optimization methods: example

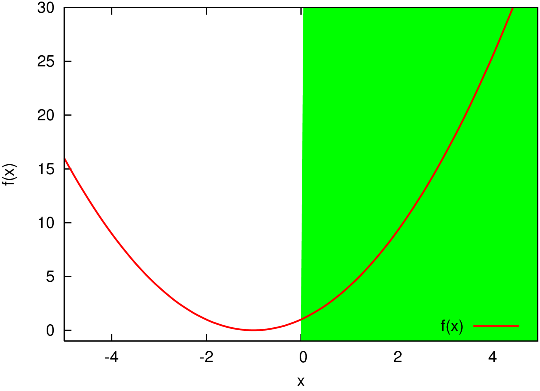


Functional : $f(x) = x^2 + 2x + 1$

Constrain : $g(x) = x \geq 0$

Solution : $x^* = 0$

Optimization methods: example



Functional : $f(x) = x^2 + 2x + 1$

Constrain : $g(x) = x \geq 0$

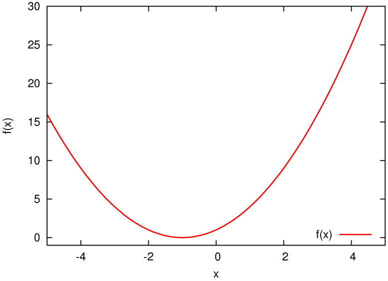
Solution : $x^* = 0$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$

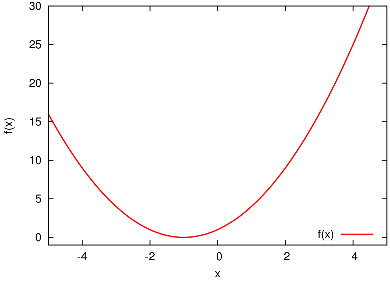


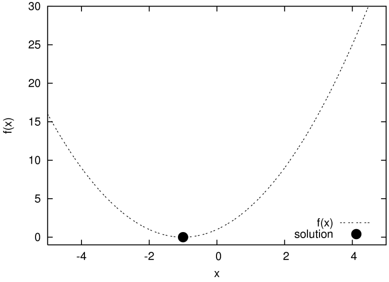
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





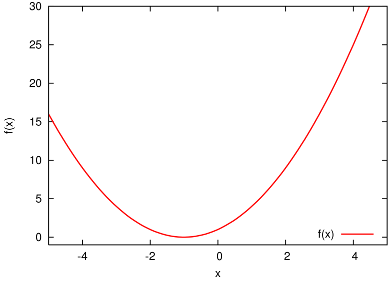
$\epsilon = 0$

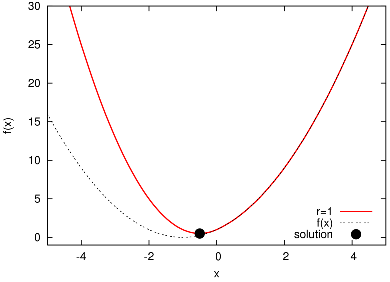
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





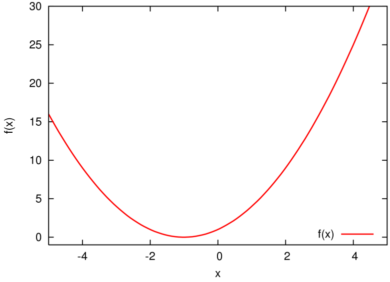
$\epsilon = 1$

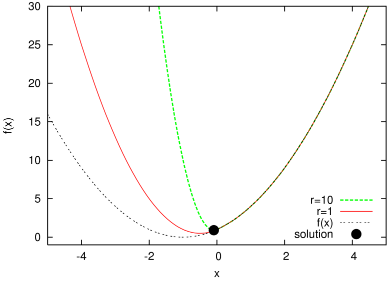
Penalty method: example

$$F(x) = x^2 + 2x + 1, \;\; g(x) = x \geq 0, \;\; x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





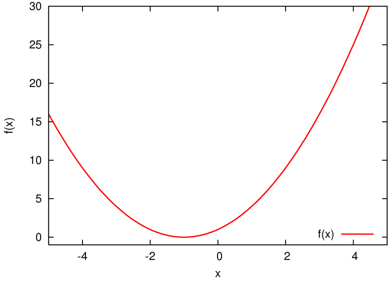
$\epsilon = 10$

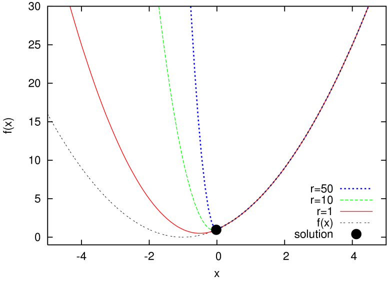
Penalty method: example

$$F(x) = x^2 + 2x + 1, \;\; g(x) = x \geq 0, \;\; x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





$\epsilon = 50$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$

Advantages ☺

- simple physical interpretation
- simple implementation
- no additional degrees of freedom
- "mathematically" smooth functional

Drawbacks ☹

- practically non-smooth functional
- solution is not exact:
  - too small penalty → large penetration
  - too large penalty → ill-conditioning of the tangent matrix
- user has to choose penalty ϵ properly or automatically and/or adapt during convergence

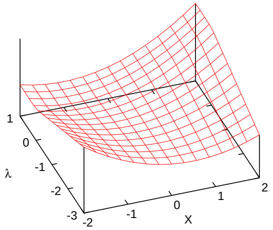
Lagrange multipliers method: example

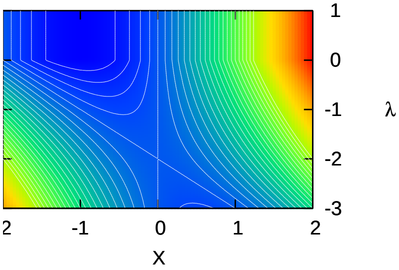
$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Lagrange multipliers method**

  $\mathcal{L}(x,\lambda) = F(x) + \boxed{\lambda g(x)} \rightarrow$ Saddle point $\rightarrow \min\limits_{x} \max\limits_{\lambda} \mathcal{L}(x,\lambda)$

  Need to check that $\lambda \leq 0$





Lagrange multipliers method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Lagrange multipliers method**

  $\mathcal{L}(x, \lambda) = F(x) + \boxed{\lambda g(x)}$ → Saddle point → $\min_x \max_\lambda \mathcal{L}(x, \lambda)$

  Need to check that $\lambda \leq 0$

Advantages ☺

- exact solution
- no adjustable parameters

Drawbacks ☹

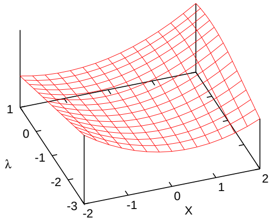
- Lagrangian is not smooth
- additional degrees of freedom
- not fully unconstrained: $\lambda \leq 0$

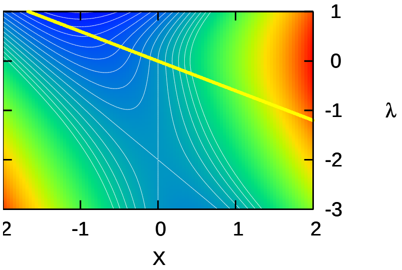
Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \ \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \ \textit{non-contact} \end{cases}$$





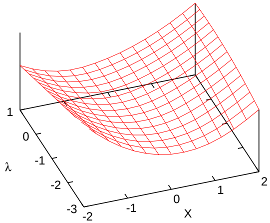
Yellow line separates contact and non-contact regions

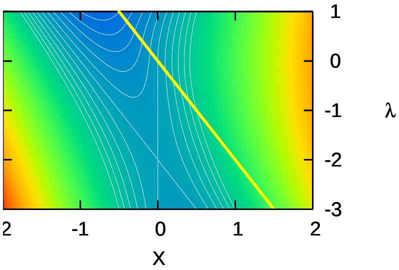
Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \textit{ contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \textit{ non-contact} \end{cases}$$





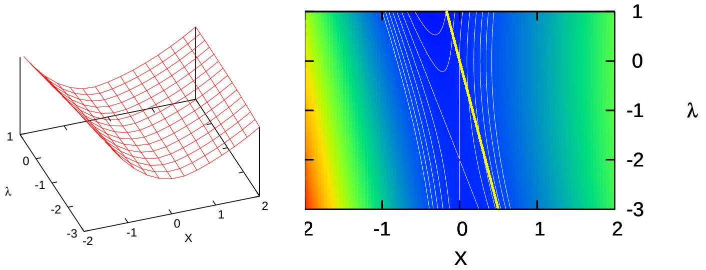
Yellow line separates contact and non-contact regions

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \textit{ contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \textit{ non-contact} \end{cases}$$



Yellow line separates contact and non-contact regions

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \textit{ contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \textit{ non-contact} \end{cases}$$

Advantages ☺

- exact solution
- smooth functional (!)
- fully unconstrained

Drawbacks ☹

- additional degrees of freedom
- quite sensitive to parameter ϵ
- need to adjust ϵ during convergence

Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} \, d\Omega + \int_{\Gamma_c^1} \underbrace{\boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \text{ on } \Gamma_u \right\}$$

- Penalty method

Pressure: $\sigma_n = \epsilon g_n$, Shear: $\underline{\sigma}_t = \begin{cases} \epsilon \underline{g}_t, & \textit{if stick } |\sigma_t| < \mu|\sigma_n| \\ \mu \epsilon g_n \delta \underline{g}_t / |\delta \underline{g}_t|, & \textit{if slip } |\sigma_t| = \mu|\sigma_n| \end{cases}$

Contact term

$$C = C(g_n, \underline{g}_t, \delta g_n, \delta \underline{g}_t) = \sigma_n \delta g_n + \underline{\sigma}_t \cdot \delta \underline{g}_t$$

Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega + \int_{\Gamma_c^1} \underbrace{\boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega,$$

$$\mathbb{K} = \left\{ \delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \;\; \text{on } \Gamma_u \right\}$$

- Augmented Lagrangian method

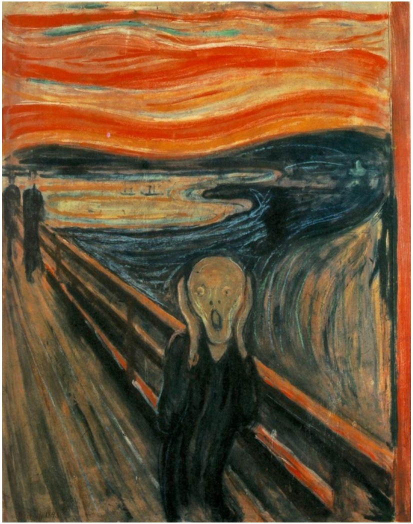
Contact term

$$C = C(g_n, \underline{g}_t, \lambda_n, \underline{\lambda}_t, \delta g_n, \delta\underline{g}_t, \delta\lambda_n, \delta\underline{\lambda}_t)$$

$$C = \begin{cases} -\frac{1}{\epsilon}\left(\lambda_n\delta\lambda_n - \underline{\lambda}_t \cdot \delta\underline{\lambda}_t\right), & \textit{if non-contact } \lambda_n + \epsilon g_n \geq 0 \\ \hat{\lambda}_n\delta g_n + g_n\delta\lambda_n + \underline{\hat{\lambda}_t} \cdot \delta\underline{g}_t + \underline{g}_t \cdot \delta\underline{\hat{\lambda}_t}, & \textit{if stick } |\underline{\hat{\lambda}}_t| \leq \mu|\hat{\sigma}_n| \\ \hat{\lambda}_n\delta g_n + g_n\delta\lambda_n + \mu\hat{\sigma}_n - \mu\hat{\sigma}_n \frac{\underline{\hat{\lambda}_t}}{|\underline{\hat{\lambda}_t}|} \cdot \delta\underline{g}_t - \frac{1}{\epsilon}\left(\lambda_t + \mu\hat{\sigma}_n \frac{\underline{\hat{\lambda}_t}}{|\underline{\hat{\lambda}_t}|}\right) \cdot \delta\underline{\lambda}_t, & \textit{if slip } |\underline{\hat{\lambda}}_t| \geq \mu|\hat{\sigma}_n| \end{cases}$$

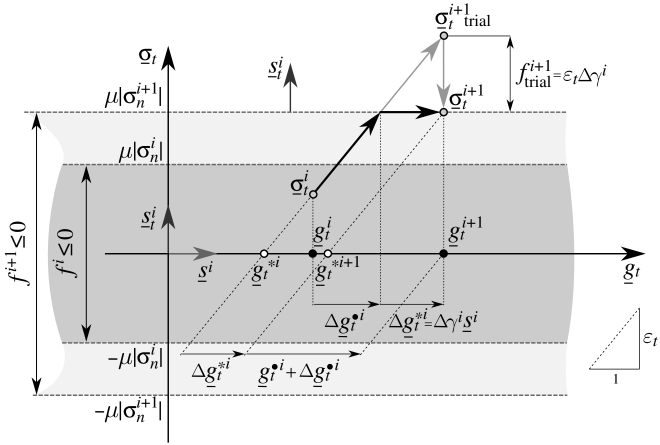
where $\hat{\lambda}_n = \lambda_n + \epsilon g_n$ and $\underline{\hat{\lambda}}_t = \underline{\lambda}_t + \epsilon \underline{g}_t$.

Friction . . . . . .



The scream

- Return mapping algorithm in 2D

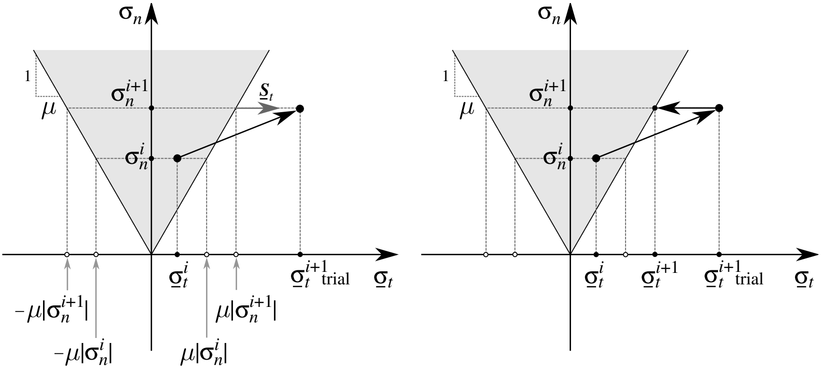


As in plasticity[1]

[1] Simo J.C. and Hughes T.J.. Computational inelasticity. Springer (2006)

Friction: Return mapping algorithm

- Return mapping algorithm in 2D

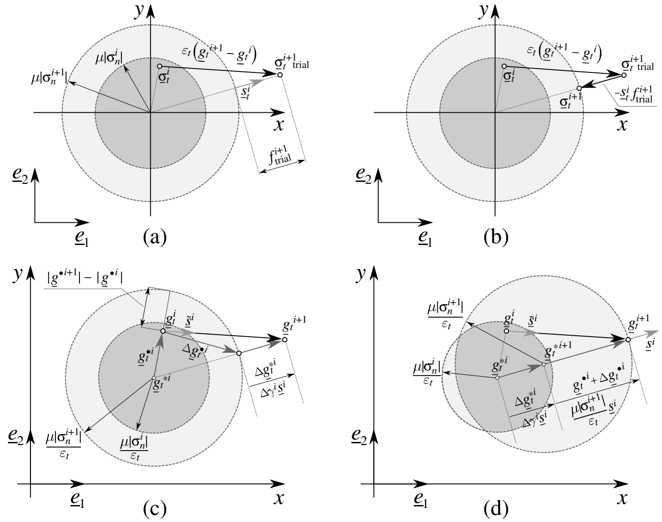


Analogy with non-associated plastic flow[2]

[2] Curnier A. A theory of friction. International Journal of Solids and Structures 20 (1984)

Friction: Return mapping algorithm

- Return mapping algorithm in 3D



A sligthly more messy thing

Application to contact problems: linearization

- Non-linear equation

$$R(\underline{\boldsymbol{u}}, \underline{\boldsymbol{f}}) = 0$$

- Contains $\delta g_n, \delta \underline{\boldsymbol{g}}_t$
- Use Newton-Raphson method
- Initial state at step i

$$R(\underline{\boldsymbol{u}}^i, \underline{\boldsymbol{f}}^i) = 0$$

- Should be also satisfied at step $i + 1$

$$R(\underline{\boldsymbol{u}}^{i+1}, \underline{\boldsymbol{f}}^{i+1}) = R(\underline{\boldsymbol{u}}^i + \delta\underline{\boldsymbol{u}}, \underline{\boldsymbol{f}}^{i+1}) = 0$$

- Linearize

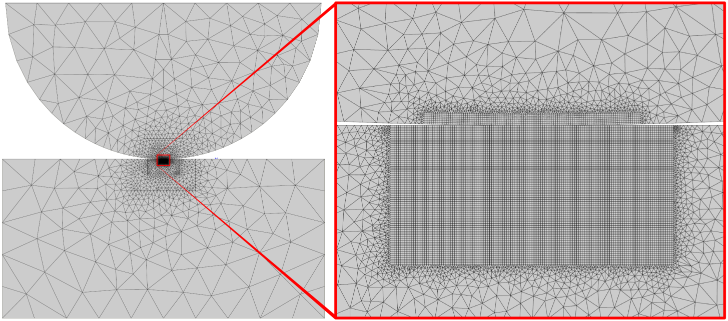
$$R(\underline{\boldsymbol{u}}^i + \delta\underline{\boldsymbol{u}}, \underline{\boldsymbol{f}}^{i+1}) = R(\underline{\boldsymbol{u}}^i, \underline{\boldsymbol{f}}^{i+1}) + \frac{\partial R(\underline{\boldsymbol{u}})}{\partial \underline{\boldsymbol{u}}} \delta\underline{\boldsymbol{u}} = 0$$

- Finally

$$\delta\underline{\boldsymbol{u}} = - \underbrace{\left[\frac{\partial R(\underline{\boldsymbol{u}})}{\partial \underline{\boldsymbol{u}}}\right]^{-1}}_{\text{contains } \Delta\delta g_n, \Delta\delta \underline{\boldsymbol{g}}_t} R(\underline{\boldsymbol{u}}^i)$$

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones

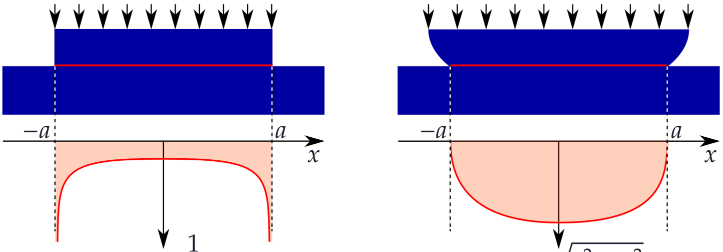


Typical mesh for fretting analysis [L. Sun, H. Proudhon, G. Cailletaud, 2011]

2D ∼ 30 000 DoFs,   3D ∼ 5 000 000 DoFs

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones



$$\sigma_n \sim \frac{1}{\sqrt{a^2 - x^2}} \qquad \sigma_n \sim \sqrt{a^2 - x^2}$$

$$\sigma_n \xrightarrow[x\to a]{} -\infty \quad \left|\frac{\partial \sigma_n}{\partial x}\right| \xrightarrow[x\to a]{} \infty \qquad \left|\frac{\partial \sigma_n}{\partial x}\right| \xrightarrow[x\to a]{} \infty$$

Infinite contact pressure and/or its derivative

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
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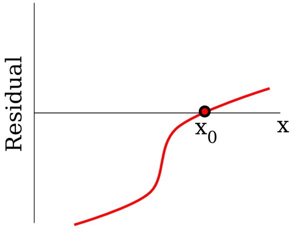
Infinite looping



Initial guess $R(x_0, f_0) = 0$

Particularities: mesh and convergence

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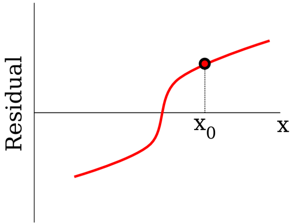

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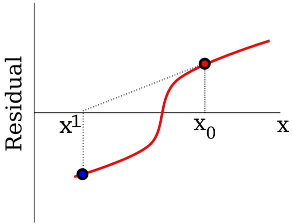

Iterations of Newton-Raphson method

$$R(x_0,f_1) + \left.\frac{\partial R}{\partial x}\right|_{x_0} \delta x = 0 \rightarrow \delta x = -\left.\frac{\partial R}{\partial x}\right|_{x_0}^{-1} R(x_0,f_1) \rightarrow x^1 = x_0 + \delta x$$

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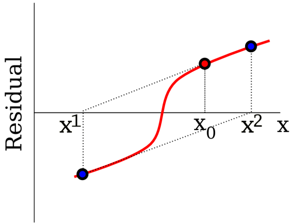

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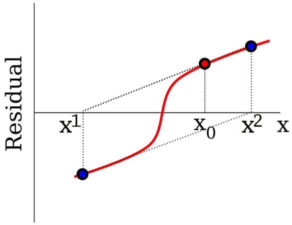

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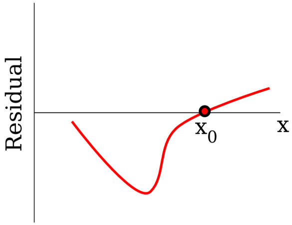

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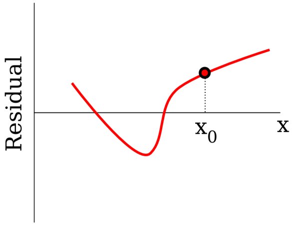

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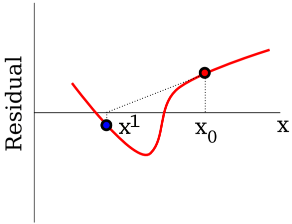

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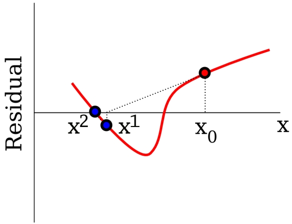

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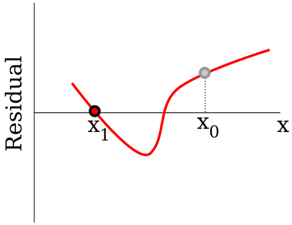
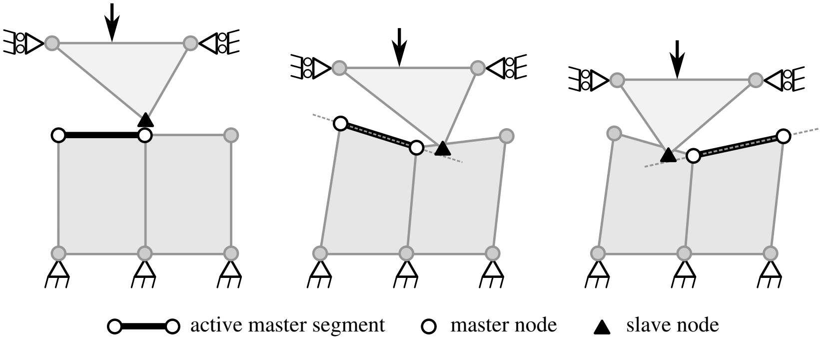


Convergence, but is it a "true" solution ?

Convergence problems: examples

- **Infinite looping, e.g.**

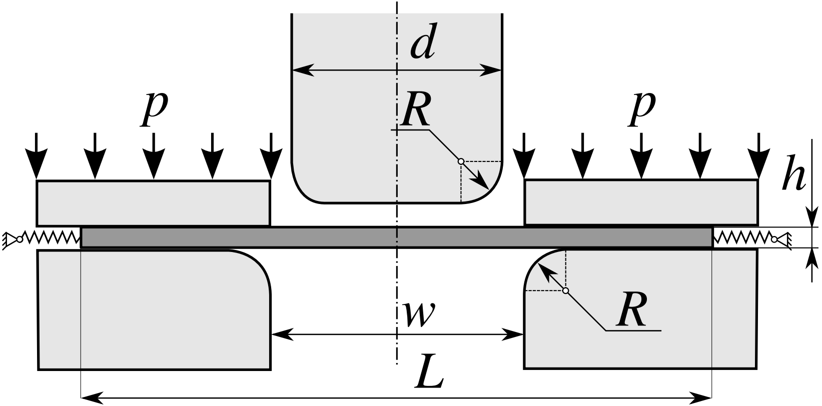


- Change of the contact state (contact/non-contact, stick/slip)
- Interplay between stiffness, friction and augmented Lagrangian coefficients[1]
- Combination of non-linearities (e.g., plasticity+contact)

Alart P., Méthode de Newton généralisée en mécanique du contact Journal de Mathématiques Pures et Appliqués 76 (1997)

Convergence problems: examples

- Simulation of a deep drawing problem
- Dinite strain plasticity + frictional contact



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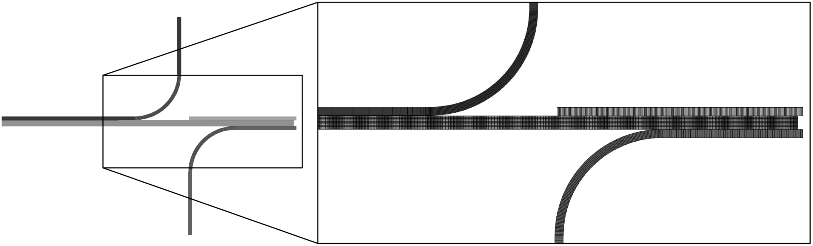

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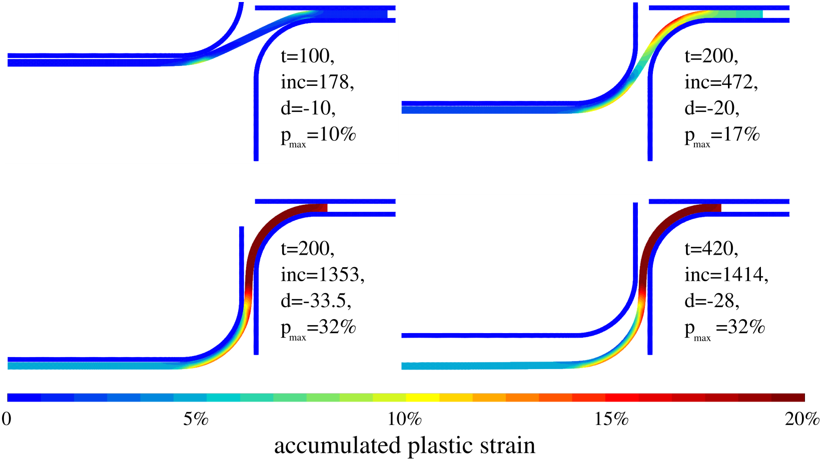

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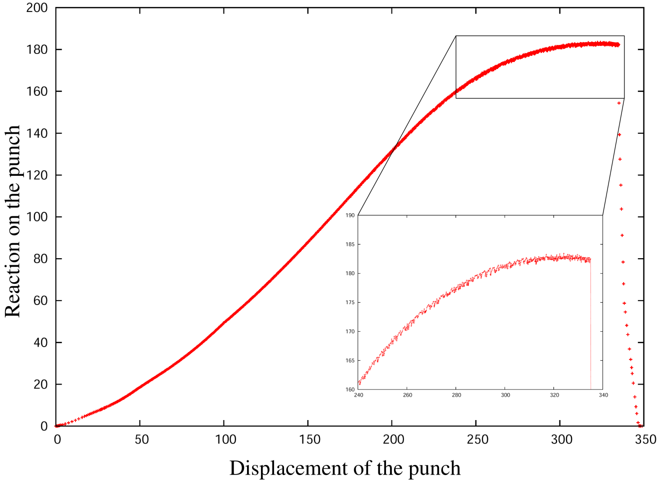


Convergence problems: examples

- Convergence?

ContactMechanics_in_Zset.pdf, page 13

Examples of contact problems

With analytical solution

- ⋆ linear elasticity
- ⋆ with/without friction

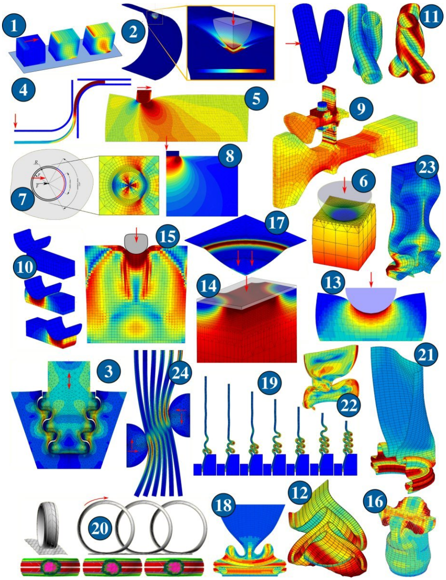
From literature

- ⋆ post-buckling 2D
- ⋆ finite strains
- ⋆ elasticity / plasticity
- ⋆ with/without friction

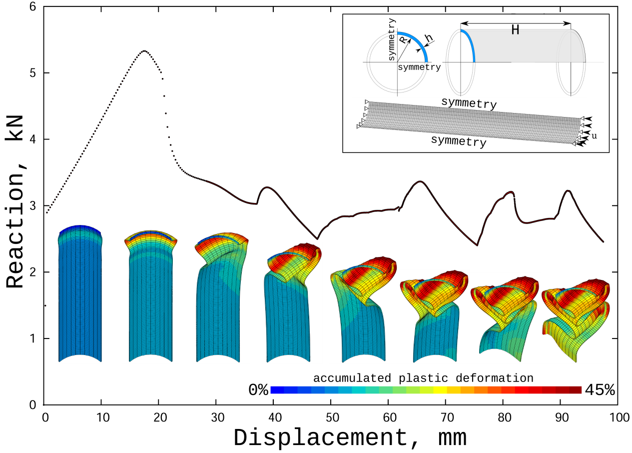
New

- ⋆ multi-contacts
- ⋆ post-buckling 3D
- ⋆ finite strains
- ⋆ elasticity / plasticity
- ⋆ with/without friction

Z-set | Material & structure analysis suite



Self-contact problem



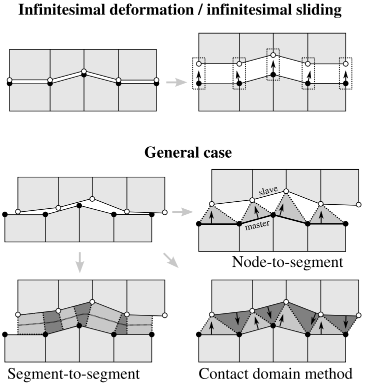
Finite element analysis of a post-buckling behavior of a thin walled tube

Collection of non-linearities: buckling instability, self-contact, finite strain plasticity

Z-set Material & structure analysis suite

Reading

- It's just a tip of the "Computational Contact Mechanics" iceberg
- Contact detection
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics



Contact discretization techniques

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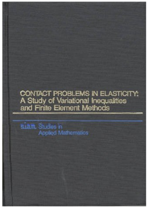
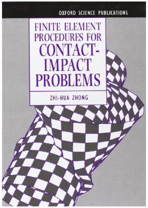
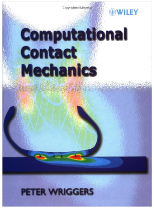


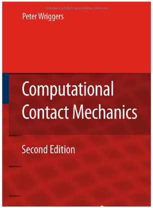
Kikuchi, Oden (1988)



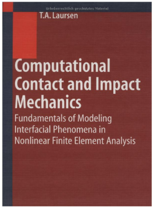
Zhong (1993)



Wriggers (2002)



Wriggers, 2nd ed. (2006)



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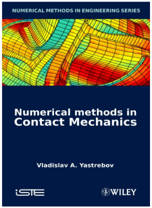
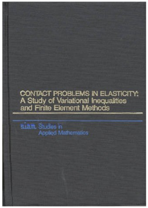


Yastrebov (2013)

Reading

- It’s just a tip of the “Computational Contact Mechanics” iceberg
- Contact detection
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics
- Several advanced topics

see `Yastrebov_CEMEF.pdf`, page 6.



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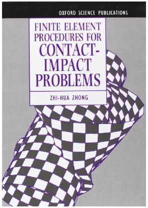

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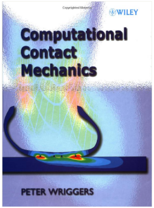

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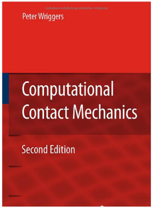

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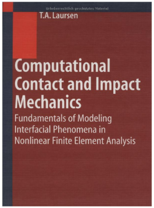

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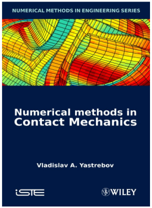

Yastrebov (2013)

Thank you for your attention!