Contact Mechanics and Elements of Tribology

Lectures 2-3.
Mechanical Contact

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@ Centre des Matériaux
February 8-9, 2016

Outline

Lecture 2

1. Balance equations
2. Intuitive notions
3. Formalization of frictionless contact
4. Evidence friction
5. Contact types
6. Analogy with boundary conditions

Lecture 3

1. Flamant, Boussinesq, Cerruti
2. Displacements and tractions
3. Classical elastic problems

Boundary value problem in elasticity

- Reference and current configurations

$$\underline{x} = \underline{X} + \underline{u}$$

- Balance equation (strong form)

$$\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{f}_v = 0, \forall \underline{x} \in \Omega^i$$

- Displacement compatibility

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla \underline{u} + \underline{u}\nabla)$$

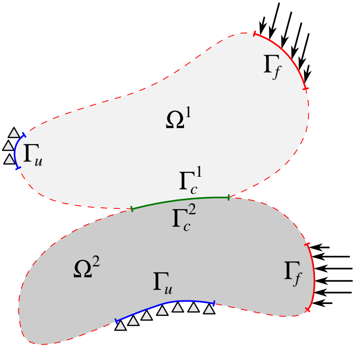
- Constitutive equation

$$\underline{\underline{\sigma}} = W'(\underline{\underline{\varepsilon}})$$

- Boundary conditions

Dirichlet: $\underline{u} = \underline{u}^0, \forall \underline{x} \in \Gamma_u$

Neumann: $\underline{n} \cdot \underline{\underline{\sigma}} = \underline{t}^0, \forall \underline{x} \in \Gamma_f$



Two bodies in contact

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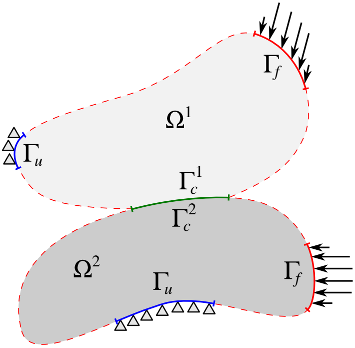


Two bodies in contact

- **Include contact conditions**

. . .

1 No penetration

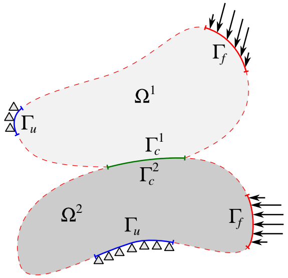
$$\Omega^1(t) \cap \Omega^2(t) = \emptyset$$

2 No adhesion

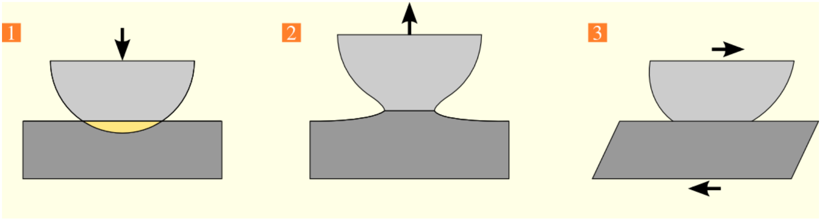
$$\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \leq 0, \forall \underline{x} \in \Gamma_c^i$$

3 No shear stress

$$\underline{n} \cdot \underline{\underline{\sigma}} \cdot (\underline{\underline{I}} - \underline{n} \otimes \underline{n}) = 0, \forall \underline{x} \in \Gamma_c^i$$



Two bodies in contact



Intuitive contact conditions for frictionless and nonadhesive contact

Intuitive conditions

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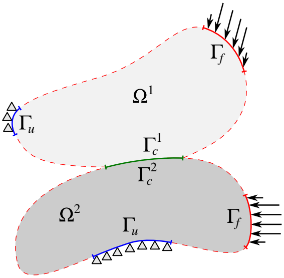
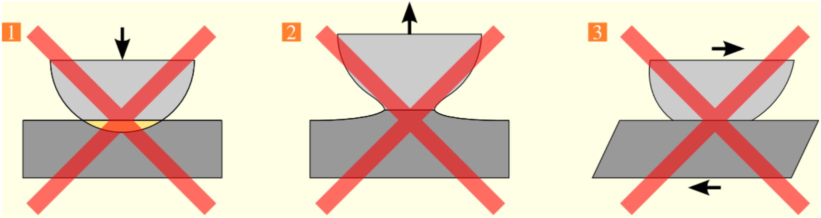


Two bodies in contact

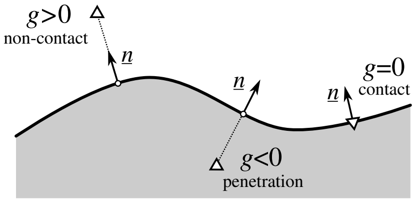


Intuitive contact conditions for frictionless and nonadhesive contact

Gap function

- **Gap function** g
  - gap = – penetration
  - asymmetric function
  - defined for
    - separation $g > 0$
    - contact $g = 0$
    - penetration $g < 0$
  - governs normal contact
- **Master and slave split**

  Gap function is determined for all slave points with respect to the master surface



Gap between a slave point and a master surface

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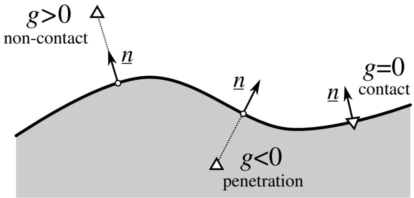

Gap between a slave point and a master surface

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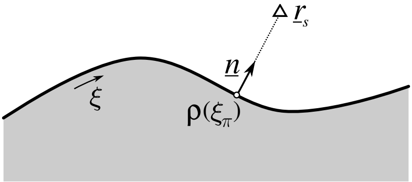


Definition of the normal gap

- **Normal gap**

$$g_n = \underline{n} \cdot \left[\underline{r}_s - \underline{\rho}(\xi_\pi) \right],$$

$\underline{n}$ is a unit normal vector, $\underline{r}_s$ slave point, $\underline{\rho}(\xi_\pi)$ projection point at master surface

Frictionless or normal contact conditions

- **No penetration**
  Always non-negative gap

  $$g \ge 0$$

- **No adhesion**
  Always non-positive contact pressure

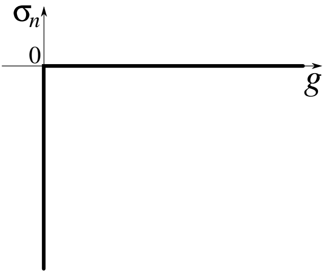
  $$\sigma_n^* \le 0$$

- **Complementary condition**
  Either zero gap and non-zero pressure, or non-zero gap and zero pressure

  $$g\,\sigma_n = 0$$

- **No shear transfer** *(automatically)*

  $$\underline{\sigma}_t^{**} = 0$$



Scheme explaining normal contact conditions

$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$

$\underline{\sigma}_t^{**} = \underline{\underline{\sigma}} \cdot \underline{n} - \sigma_n \underline{n} = \underline{n} \cdot \underline{\underline{\sigma}} \cdot \left(\underline{\underline{I}} - \underline{n} \otimes \underline{n}\right)$

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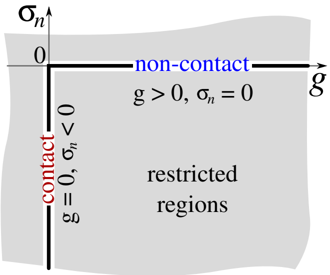

Improved scheme explaining normal contact conditions

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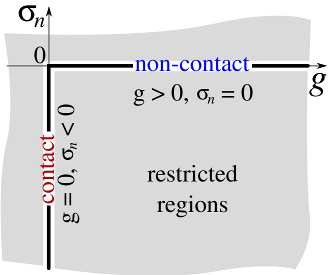
Frictionless or normal contact conditions

In **mechanics:**

Normal contact conditions
≡
Frictionless contact conditions
≡
Hertz[1]*-Signorini*[2] *conditions*
≡
Hertz[1]*-Signorini*[2]*-Moreau*[3] *conditions*

also known in **optimization theory** as

Karush[4]*-Kuhn*[5]*-Tucker*[6] *conditions*



Improved scheme explaining normal contact conditions

$$g \geq 0, \qquad \sigma_n \leq 0, \qquad g\sigma_n = 0$$

[1] **Heinrich Rudolf Hertz** (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.
[2] **Antonio Signorini** (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.
[3] **Jean Jacques Moreau** (1923) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.
[4] **William Karush** (1917–1997), [5] **Harold William Kuhn** (1925) American mathematicians,
[6] **Albert William Tucker** (1905–1995) a Canadian mathematician.

Contact problem

$\approx$ **Problem**

Find such contact pressure

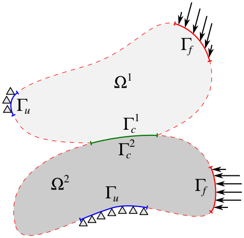
$$p = -\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \geq 0$$

which being applied at Γ_c^1 and Γ_c^2 results in

$$\underline{x}^1 = \underline{x}^2, \forall \underline{x}^1 \in \Gamma_c^1, \underline{x}^2 \in \Gamma_c^2$$

and evidently

$$\Omega^1(t) \cap \Omega^2(t) = \emptyset$$



Two bodies in contact

- Unfortunately, we do not know Γ_c^1 in advance, it is also an unknown of the problem.
- **Related problem**
  Suppose that we know p on Γ_c
  Then what is the corresponding displacement field $\underline{u}$ in Ω^i?

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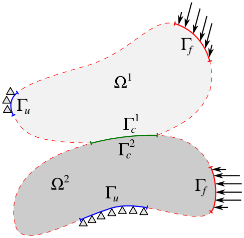

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  Suppose that we know p on Γ_c
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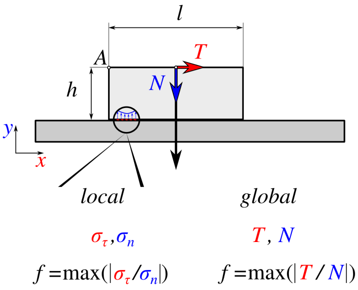
   This afternoon

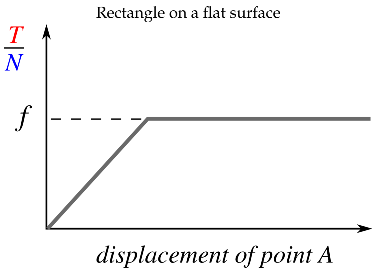
Evidence of friction

- Existence of frictional resistance is evident
- Independence of the nominal contact area

Think about adhesion and introduce a threshold in the interface τ_c

- Globally:
  - stick: $T < T_c(N)$
  - slip: $T = T_c(N)$
- From experiments:
  - Threshold $T_c \sim N$
  - Friction coefficient $f = |N/T_c|$
- Locally
  - stick: $\sigma_\tau < \tau_c(\sigma_n)$
  - slip: $\sigma_\tau = f\sigma_n$





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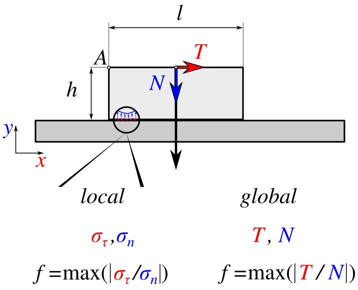
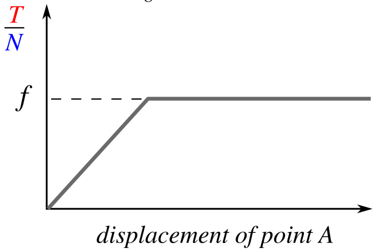
- Torque

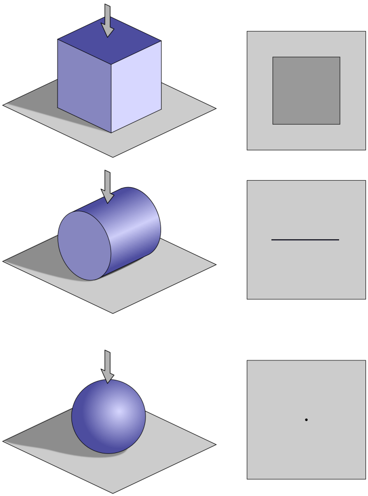




Types of contact

- Known contact zone
  - conformal geometry
    flat-to-flat, cylinder in a hole
  - initially non-conformal geometry but huge pressure resulting in full contact
- Unknown contact zone
  general case
- Point and line contact
- Frictionless
  conservative, energy minimization problem
- Frictional
  path-dependent solution, from the first touch to the current moment

Example



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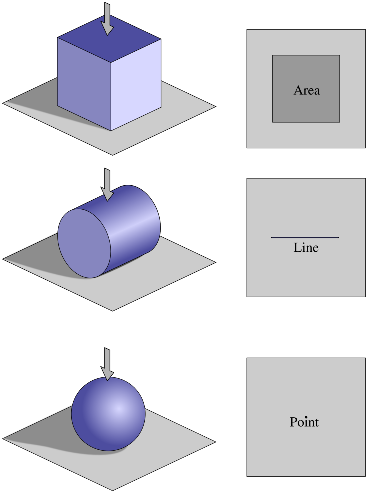

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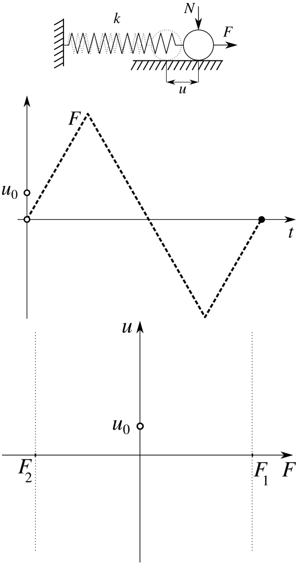
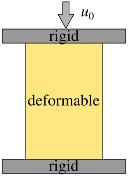
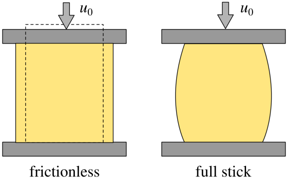
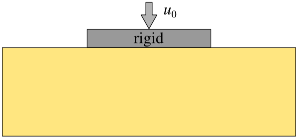


Flat geometry

- Compression of a cylinder
- Frictionless $u_z = u_0$
- Full stick conditions $\underline{\boldsymbol{u}} = u_0\underline{\boldsymbol{e}}_z$
- Rigid flat indenter $u_z = u_0$







Analogy with boundary conditions

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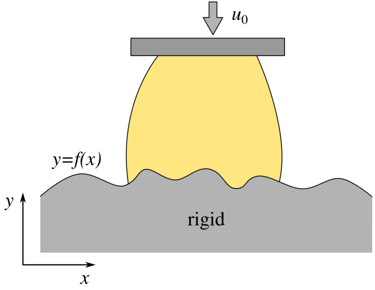
Curved geometry

- Polar/spherical coordinates $u_r = u_0$
- If frictionless contact on rigid surface $y = f(x)$ is retained by high pressure

$$(\underline{\boldsymbol{X}} + \underline{\boldsymbol{u}}) \cdot \underline{\boldsymbol{e}}_y = f((\underline{\boldsymbol{X}} + \underline{\boldsymbol{u}}) \cdot \underline{\boldsymbol{e}}_x)$$



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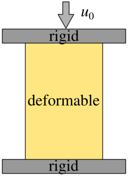
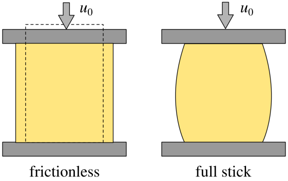
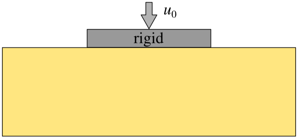
Transition to finite friction

-  ≈ From full stick, decrease f by keeping $u_z = 0$ and by replacing in-plane Dirichlet BC by in-plane Neumann BC







Analogy with boundary conditions II

In general

- Type I: prescribed tractions 
  $p(x,y), \tau_x(x,y), \tau_y(x,y)$
- Type II: prescribed displacements
  $\underline{\boldsymbol{u}}(x,y)$
- Type III: tractions and displacements
  $u_z(x,y), \tau_x(x,y), \tau_y(x,y)$ or
  $p(x,y), u_x(x,y), u_y(x,y)$
- Type IV: displacements and relation between tractions
  $u_z(x,y), \tau_x(x,y) = \pm fp(x,y)$

To be continued. . .

Concentrated forces

- Normal force: in-plane stresses and displacements (plane strain)

$$\sigma_r = -\frac{2N}{\pi}\frac{\cos(\theta)}{r} \text{ or } \sigma_x = -\frac{2N}{\pi}\frac{x^2y}{(x^2+y^2)^2},\ \sigma_y = -\frac{2N}{\pi}\frac{y^3}{(x^2+y^2)^2},\ \sigma_{xy} = -\frac{2N}{\pi}\frac{xy^2}{(x^2+y^2)^2}$$

$$u_r = \frac{1+\nu}{\pi E} N\cos(\theta)\left[2(1-\nu)\ln(r) - (1-2\nu)\theta\tan(\theta)\right] + C\cos(\theta)$$

$$u_\theta = \frac{1+\nu}{\pi E} N\sin(\theta)\left[2(1-\nu)\ln(r) - 2\nu + (1-2\nu)(1-2\theta\,\text{ctan}(\theta))\right] - C\sin(\theta)$$

- Tangential force

$$\sigma_r = \frac{2T}{\pi}\frac{\sin(\theta)}{r} \text{ or } \sigma_x = -\frac{2T}{\pi}\frac{x^3}{(x^2+y^2)^2},\ \sigma_y = -\frac{2T}{\pi}\frac{xy^2}{(x^2+y^2)^2},\ \sigma_{xy} = -\frac{2T}{\pi}\frac{x^2y}{(x^2+y^2)^2}$$

$$u_r = -\frac{1+\nu}{\pi E} T\sin(\theta)\left[2(1-\nu)\ln(r) - (1-2\nu)\theta\text{ctan}(\theta)\right] - C\sin(\theta)$$

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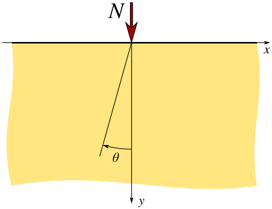
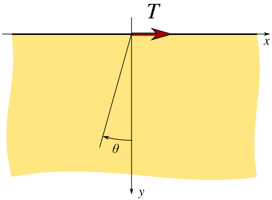
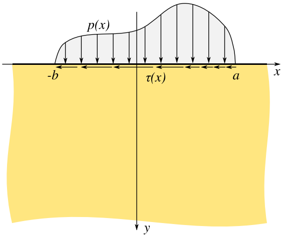




Distributed load

- Distributed tractions $p(x)dx = dN(x)$, $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements



Tractions on the surface

$$\sigma_x(x,y) = -\frac{2y}{\pi}\int_{-b}^{a}\frac{p(s)(x-s)^2\,ds}{((x-s)^2+y^2)^2} - \frac{2}{\pi}\int_{-b}^{a}\frac{\tau(s)(x-s)^3\,ds}{((x-s)^2+y^2)^2}$$

$$\sigma_y(x,y) = -\frac{2y^3}{\pi}\int_{-b}^{a}\frac{p(s)\,ds}{((x-s)^2+y^2)^2} - \frac{2y^2}{\pi}\int_{-b}^{a}\frac{\tau(s)(x-s)\,ds}{((x-s)^2+y^2)^2}$$

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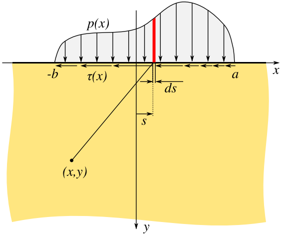


Tractions on the surface

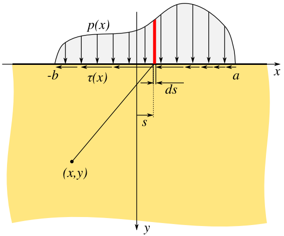
$$\sigma_x(x,y) = -\frac{2y}{\pi}\int_{-b}^{a}\frac{p(s)(x-s)^2\,ds}{((x-s)^2+y^2)^2} - \frac{2}{\pi}\int_{-b}^{a}\frac{\tau(s)(x-s)^3\,ds}{((x-s)^2+y^2)^2}$$

$$\sigma_y(x,y) = -\frac{2y^3}{\pi}\int_{-b}^{a}\frac{p(s)\,ds}{((x-s)^2+y^2)^2} - \frac{2y^2}{\pi}\int_{-b}^{a}\frac{\tau(s)(x-s)\,ds}{((x-s)^2+y^2)^2}$$

$$\sigma_{xy}(x,y) = -\frac{2y^2}{\pi}\int_{-b}^{a}\frac{p(s)(x-s)\,ds}{((x-s)^2+y^2)^2} - \frac{2y}{\pi}\int_{-b}^{a}\frac{\tau(s)(x-s)^2\,ds}{((x-s)^2+y^2)^2}$$

Distributed load

- Distributed tractions $p(x)dx = dN(x)$, $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements
- Consider displacements on the surface

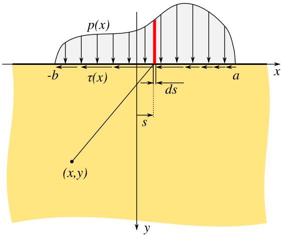


Tractions on the surface

$$u_x(x,0) = -\frac{(1-2\nu)(1+\nu)}{2E}\left[\int_{-b}^{x} p(s)\,ds - \int_{x}^{a} p(s)\,ds\right] - \frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a} \tau(s)\ln|x-s|\,ds + C_1$$

Distributed load

- Distributed tractions $p(x)dx = dN(x)$, $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements
- Consider displacements on the surface
- Or rather their derivatives along the surface



Tractions on the surface

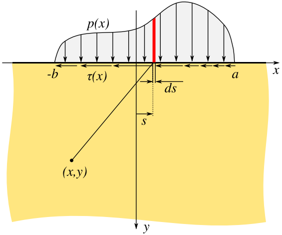
$$u_x(x,0) = -\frac{(1-2\nu)(1+\nu)}{2E}\left[\int_{-b}^{x} p(s)\,ds - \int_{x}^{a} p(s)\,ds\right] - \frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a} \tau(s)\ln|x-s|\,ds + C_1$$

$$u_{x,x}(x,0) = -\frac{(1-2\nu)(1+\nu)}{E}p(x) - \frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a}\frac{\tau(s)}{x-s}\,ds$$

Near-surface stress state

Distributed load

- Distributed tractions $p(x)dx = dN(x)$, $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements
- Consider displacements on the surface
- Or rather their derivatives along the surface

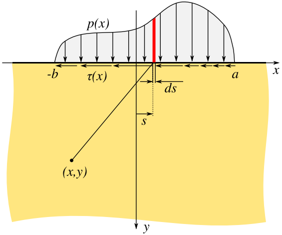


Tractions on the surface

$$u_y(x,0) = \frac{(1-2\nu)(1+\nu)}{2E}\left[\int_{-b}^{x}\tau(s)\,ds - \int_{x}^{a}\tau(s)\,ds\right] - \frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a} p(s)\ln|x-s|\,ds + C_2$$

Distributed load

- Distributed tractions $p(x)dx = dN(x)$, $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements
- Consider displacements on the surface
- Or rather their derivatives along the surface



Tractions on the surface

$$u_y(x,0) = \frac{(1-2\nu)(1+\nu)}{2E}\left[\int_{-b}^{x} \tau(s)\,ds - \int_{x}^{a} \tau(s)\,ds\right] - \frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a} p(s)\ln|x-s|\,ds + C_2$$

$$u_{y,x}(x,0) = \frac{(1-2\nu)(1+\nu)}{E}\tau(x) - \frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a} \frac{p(s)}{x-s}\,ds$$

Rigid stamp problem

- Link displacement derivatives with tractions

$$\int_{-b}^{a} \frac{\tau(s)}{x-s}\,ds = -\frac{\pi(1-2\nu)}{2(1-\nu)}p(x) - \frac{\pi E}{2(1-\nu^2)}u_{x,x}(x,0)$$

$$\int_{-b}^{a} \frac{p(s)}{x-s}\,ds = \frac{\pi(1-2\nu)}{2(1-\nu)}\tau(x) - \frac{\pi E}{2(1-\nu^2)}u_{y,x}(x,0)$$

- If in contact interface we can prescribe $p, u_{x,x}$ or $\tau, u_{y,x}$, then the problem reduces to

$$\int_{-b}^{a} \frac{\mathcal{F}(x)}{x-s}ds = \mathcal{U}(x)$$

- The general solution (case $a = b$):

$$\mathcal{F}(x) = \frac{1}{\pi^2\sqrt{a^2-x^2}} \int_{-a}^{a} \frac{\sqrt{a^2-x^2}\,\mathcal{U}(s)\,ds}{x-s} + \frac{C}{\pi\sqrt{a^2-x^2}}, \quad C = \int_{-a}^{a} \mathcal{F}(s)ds$$

Rigid stamp problem

- Link displacement derivatives with tractions

$$\int_{-b}^{a} \frac{\tau(s)}{x-s}\,ds = -\frac{\pi(1-2\nu)}{2(1-\nu)}p(x) - \frac{\pi E}{2(1-\nu^2)}u_{x,x}(x,0)$$

$$\int_{-b}^{a} \frac{p(s)}{x-s}\,ds = \frac{\pi(1-2\nu)}{2(1-\nu)}\tau(x) - \frac{\pi E}{2(1-\nu^2)}u_{y,x}(x,0)$$

- If in contact interface we can prescribe $p, u_{x,x}$ or $\tau, u_{y,x}$, then the problem reduces to

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flat frictionless punch, consider P.V.

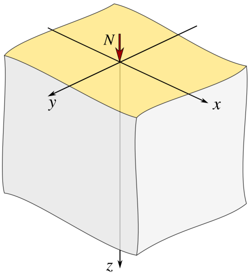
Three-dimensional problem

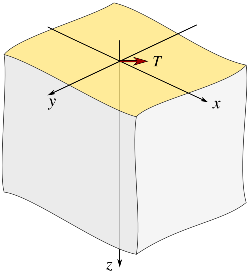
- Analogy to Flamant's problem
- Potential functions of Boussinesq
- Boussinesq problem
  concentrated normal force
- Cerruti problem
  concentrated tangential force
- Displacements decay as $\sim r^{-1}$

$$u_r(x,y,0) = -\frac{1-2\nu}{4\pi G}\frac{N}{\sqrt{x^2+y^2}}$$

$$u_z(x,y,0) = \frac{1-\nu}{4\pi G}\frac{N}{\sqrt{x^2+y^2}}$$

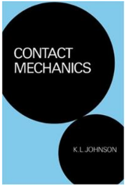
- Stress decay as $\sim r^{-2}$
- Superposition principle





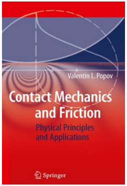
Classical contact problems

- Various problems with rigid flat stamps: *circular, elliptic, frictionless, full-stick, finite friction*
- Hertz theory
  normal frictionless contact of elastic solids 
  E_i, ν_i and $z_i = A_i x^2 + B_i y^2 + C_i xy, \quad i = 1,2$
- Wedges *(coin)* and cones
- Circular inclusion in a conforming hole
  Steuermann, 1939,Goodman, Keer, 1965
- Frictional indentation $z \sim x^n$
  Incremental approach Mossakovski, 1954
  self-similar solution Spence, 1968, 1975
- Adhesive contact Johnson et al, 1971, 1976
- Contact with layered materials (coatings)
- Elastic-plastic and viscoelastic materials
- Sliding/rolling of non-conforming bodies
  Cattaneo, 1938,Mindlin, 1949,Galin, 1953,Goryacheva, 1998
  Note: $u_r \sim (1-2\nu)/G$, so if $(1-2\nu_1)/G_1 = (1-2\nu_2)/G_2$ tangential tractions do not change normal ones



K.L. Johnson

I.G. Goryacheva



V.L. Popov

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